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ON U-RANK 2 TYPES

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ABSTRACT. Let T be a superstable theory with $< 2^{\aleph_0}$ countable models. We study some special types $p \in S(\varnothing)$ of U-rank 2 called skeletal (cf. [Bu4]). We reduce an eventual version of the problem of counting isomorphism types of sets $p(M)$ for countable M to a problem from linear algebra.

0. INTRODUCTION

Throughout, T is a superstable theory with $< 2^{\aleph_0}$ countable models and we work with T^{eq}. In our notation we usually follow [Ba]. For example, if $A \subseteq B$ and $p \in S(B)$ then we say that p is based on A if p does not fork over A [Ba, IV.1.17 (ii)]. For background on stability theory see [Ba, Sh]. For the definition and basic properties of modularity in the context of stability theory see [Bu1, CHL] or [P]. In [Bu4, Ne3], a study of the possible isomorphism types of sets $p(M)$ for a type p of U-rank 2 was initiated. This investigation is strongly connected with Vaught's conjecture for superstable theories of finite ∞-rank (cf. [Bu4, §0]). Following [Bu4], we call a type $p \in S(\varnothing)$ skeletal if p is stationary, has U-rank 2 and for a realizing p there is a $b \in \operatorname{acl}(a)$ such that $U(b) = 1$ and $\operatorname{tp}(a/b)$ has finite multiplicity. $q = \operatorname{stp}(b)$ is called a base type of p. In this paper we often refer to [Bu4]. There are two important contributions from [Bu4, §2] here. Firstly, [Bu4, §2] proves that $I(T, \aleph_0) < 2^{\aleph_0}$ implies q is locally modular (see Lemma 1.9 here). Secondly, it proves a local version of NOTOP (see Theorem 1.16 below).

For a type p over $\varnothing$ let $I(p, \kappa)$ denote the number of isomorphism types of sets $p(M)$, where M is a model of T of power κ. We consider the following conjecture.

(P) If T is superstable, p is skeletal and $I(T, \aleph_0) < 2^{\aleph_0}$ then $I(p, \aleph_0)$ is countable.

While a proof of (P) may not yield immediately a proof of Vaught's conjecture for superstable theories of finite rank, it may give strong indications on how to prove Vaught's conjecture in this case. Moreover, Vaught's conjecture implies (P). Here we prove some structure theorems connected with (P). We prove that for some finite set E, the problem of counting isomorphism types of sets $(p|E)(M)$ for countable M reduces to a problem from linear algebra. The

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If $\mathscr{F}$ is a family of nonempty sets, then $S \subseteq \bigcup \mathscr{F}$ is called a selector from $\mathscr{F}$ if $|S \cap X| = 1$ for every $X \in \mathscr{F}$. We work within a monster model $\mathfrak{C}$. We use $\underline{a}$, $\underline{b}$ and so on to denote tuples of elements of $\mathfrak{C}$.

For the rest of this paper, p denotes a skeletal type and q is a fixed base type of p. Let a realize p and $b \in \mathrm{acl}(a)$ realize q. Let p_b denote $\mathrm{tp}(a/b)$. So p_b has finite multiplicity and necessarily U-rank 1. For any c realizing q, let p_c be the conjugate of p_b over c. As in [Bu4, §1], q has only finitely many conjugates over $\varnothing$. Hence by adding an element of $\mathrm{acl}(\varnothing)$ to the signature we may assume that q has only one conjugate over $\varnothing$, i.e. that q is just a stationary type over $\varnothing$.

Our goal is to classify isomorphism types of countable sets of the form $p(M)$, and to prove that $I(p, \aleph_0)$ is countable. Some cases are trivial (cf. [Bu4, §1]). Eliminating these cases by [Bu4, §1] we fix the following assumptions from now on.

0.1. **Assumptions.** q is a complete stationary type over $\varnothing$. Let $Q = q(\mathfrak{C})$. For $a \in Q$, p_a is nontrivial properly weakly minimal, nonisolated and of finite multiplicity. Moreover, for b realizing p_a, $\mathrm{stp}(b/a)$ is locally modular, nonmodular, nonorthogonal to $\varnothing$ and almost orthogonal to $\varnothing$. In particular, p_a is weakly orthogonal to $q|a$.

Using these assumptions we prove the following lemma.

0.2. **Lemma.** *Let $b, c \in Q$.*

(1) *Suppose a realizes p_b and p_c. Then b, c are interalgebraic and* $\mathrm{stp}(a/b)$, $\mathrm{stp}(a/c)$ *are parallel.*

(2) *All stationarizations of p_b are nonorthogonal.*

Proof. (1) We have $b, c \in \mathrm{acl}(a)$. If $c \notin \mathrm{acl}(b)$, then we get that p_b is not almost orthogonal to $\varnothing$, a contradiction. Hence b, c are interalgebraic. It follows that $\mathrm{stp}(a/b)$, $\mathrm{stp}(a/c)$ are parallel.

(2) Let r_0, r_1 be stationarizations of p_b. Since r_i is nonorthogonal to $\varnothing$, for $d \in Q \backslash \mathrm{acl}(b)$ there is a stationarization r' of p_d nonorthogonal to r_i. It follows that for a realizing $p|b$, there are $d_0, d_1 \in \mathrm{acl}(a) \cap Q$ such that a realizes p_{d_0} and p_{d_1}, and $r'_i = \mathrm{stp}(a/d_i)$ is nonorthogonal to r_i, $i = 0, 1$. By (1), r'_0, r'_1 are parallel, hence r_0, r_1 are nonorthogonal.

Notice that if p_a is properly weakly minimal and nonisolated, then by [Ne1], $I(T, \aleph_0) < 2^{\aleph_0}$ implies that p_a has finite multiplicity. An important tool for us will be algebraic dependence ACL, introduced in [Ne2] (and denoted there by acl^*).

0.3. **Definition.** Suppose $A \subseteq \mathfrak{C}$. Let $P^\#$ be a family of properly weakly minimal locally modular pairwise nonorthogonal types over $\mathrm{acl}(A)$. We define a dependence relation ACL_A on $P^\#$. For $r \in P^\#$ and $R \subseteq P^\#$ we define $r \in \mathrm{ACL}_A(R)$ iff r is realized in the algebraic closure of $A \cup \bigcup\{r'(\mathfrak{C}) : r' \in R\}$. DIM_A denotes the ACL-dimension. When no confusion arises, we omit A in ACL_A, DIM_A. The following lemma was proved in [Ne2, 2.11] (but the proof relies on [H]).

0.4. **Lemma.** *For $R \neq \varnothing$, $r \in \mathrm{ACL}(R)$ iff r is modular or whenever B contains a realization of every $r' \in R$ then r is realized in* $\mathrm{acl}(AB)$.

0.5. **Lemma.** *Assume* $A \subseteq \mathfrak{C}$ *is countable,* $p^\#$ *is a stationary properly w.m. nontrivial type over* $\mathrm{acl}(A)$ *and* $P^\#$ *is the family of weakly minimal nonmodular types over* $\mathrm{acl}(A)$, *nonorthogonal to* $p^\#$.

(1) ACL *satisfies the exchange principle on* $P^\#$.

(2) ACL *is modular on* $P^\#$.

Proof. (1) follows by Lemma 0.4.

(2) is implicit in [Ne2, end of §2] and in [Ne1, 3.3]; the proof appears in [Ne3, 1.2] and essentially also in [Bu4, 1.11].

Assume d is a modular dependence relation on a set X, for all d-independent $x, y \in X$ there is $z \in d(x, y)\setminus(d(x) \cup d(y))$, and d-dimension of X is ≥ 4. Then after identifying d-interdependent elements of X, X with the induced dependence relation becomes a projective space over some division ring F (we say that X is projective over F). Let $P^\#$ be as in Lemma 0.5. Any $r \in P^\#$ is locally modular, hence the pregeometry on $r(\mathfrak{C})$ gives rise to a division ring F_r, and obviously for $r \neq r' \in P^\#$, F_r and $F_{r'}$ are isomorphic. We define F_v as any F_r, $r \in P^\#$. On the other hand, by Lemma 0.5, ACL is modular on $P^\#$. By [Bu2, 2.4] there is a weakly minimal $\varphi \in p^\#$ such that every w.m. q containing φ is nonorthogonal to $p^\#$. Hence $\mathrm{DIM}(P^\#)$ is infinite. Also, [Bu1, Theorem 2] yields quickly that ACL-dependence on $P^\#$ is nontrivial. Hence $P^\#$ is projective over some division ring F_h. h and v in F_h, F_v stand for "horizontal" and "vertical". In [Ne2, 6.9] we conjectured that F_h and F_v are isomorphic. Below we prove this conjecture. The proof, suggested by the referee, is much like the proof of [Ne2, 6.8], but uses 1-basedness of unidimensional theory.

0.6. **Proposition.** F_h *and* F_v *are isomorphic.*

Proof. W.l.o.g. the type $p^\#$ from Lemma 0.5 is modular and we add $\mathrm{acl}(A)$ to the signature. By [Bu2, 2.4], restricting to some $\varphi \in p^\#$, we may assume that T is weakly minimal and unidimensional, hence in particular 1-based.

We shall find an isomorphism between a projective plane over F_h in $P^\#$ and a projective plane over F_v in $p^\#(\mathfrak{C})$. Let $r_0, r_1, r_2 \in P^\#$ be ACL-independent. Let a, b, c realize r_0, r_1, r_2 respectively. Note first that if $U(e) = 1$ and $e \in \mathrm{acl}(abc)$ then $\mathrm{stp}(e)$ is nonmodular, hence belongs to $P^\#$. Let $a'b'c'$ have the same strong type as abc and be independent from abc (over $\varnothing$). We show

(*) for any $d \in \mathrm{acl}(abca'b'c')$ realizing $p^\#|a'b'c'$ there is an $e \in \mathrm{acl}(abc)$ with $\mathrm{stp}(e) \in P^\#$ and d interalgebraic with e over $a'b'c'$.

Indeed, choose e interalgebraic with $Cb(a'b'c'd/abc)$. By 1-basedness, $e \in \mathrm{acl}(abc) \cap \mathrm{acl}(a'b'c'd)$. Since $abca'b'c'$, we have $e \perp a'b'c'$. $d \in \mathrm{acl}(abca'b'c')$ implies $e \not\perp d(abca'b'c')$. We get $U(e) = 1$, and (*) follows.

Now we define a function $f\colon \mathrm{ACL}(r_0r_1r_2) \to p^\#(\mathfrak{C})$ as follows. Let $r \in \mathrm{ACL}(r_0r_1r_2)$. Then there are $e \in \mathrm{acl}(abc)$ and $e' \in \mathrm{acl}(a'b'c')$ realizing r. Clearly e and e' are independent, hence $r|e'$, and the more so $r|a'b'c'$ is modular, and e realizes $r|a'b'c'$. Since r is nonorthogonal to $p^\#$, there is a $d \in \mathrm{acl}(ea'b'c')$ realizing $p^\#|a'b'c'$. Let $d = f(r)$.

(*) shows that for every $d' \in \mathrm{acl}(abca'b'c')$ realizing $p^\#|a'b'c'$ there is a $d \in \mathrm{acl}(abca'b'c')$ realizing $p^\#|a'b'c'$ which is interalgebraic with d' over $a'b'c'$ and ii in the range of f. It follows that f induces an isomorphism between the projective plane generated by r_0, r_1, r_2 in $P^\#$ and the projective plane over $a'b'c'$ generated in $p^\#(\mathfrak{C})$ by a, b, c. In particular, F_v and F_h are isomorphic.

1. ACL and many model arguments

Now for the rest of this paper we will fix a specific meaning of $P^\#$. Let $P^\#$ be the set of weakly minimal, nonmodular types over $\operatorname{acl}(Q)$ nonorthogonal to p_a for some (every) $a \in Q$. Let $A \subseteq Q$. We define the following families of types.

$$P_A^0 = \{\operatorname{stp}(b/a)|\operatorname{acl}(A) \colon a \in \operatorname{acl}(A) \cap Q \text{ and } b \text{ realizes } p_a\},$$

$$P_A^\# = \{r|\operatorname{acl}(A) \colon r \in P^\# \text{ and } r \text{ is based on } A\},$$

$$P_A^* = \{r \in P_A^\# \colon \text{ for some finite } A' \subseteq A,\ r \text{ is based on } A' \text{ and has finitely many conjugates over } A'\},$$

$$P_A = \{r|\operatorname{acl}(A) \colon r \in \operatorname{ACL}_Q(P_Q^0) \text{ and } r \text{ is based on } A\}.$$

We write P, P^0, P^* and $P^\#$ for P_Q, P_Q^0, P_Q^* and $P_Q^\#$. Also we write P_a instead of $P_{\{a\}}$ (with all possible superscripts). For $A \subseteq B \subseteq Q$ we can naturally regard $P_A^\#$ as a subset of $P_B^\#$, identifying $r \in P_A^\#$ with $r|\operatorname{acl}(B)$ (similarly with P_A^0, P_A^* and P_A). We say that $R, R' \subseteq P^\#$ are isomorphic, if some $f \in \operatorname{Aut}(\mathfrak{C})$ induces a bijection from R onto R'. In the next lemma we collect basic properties of the families of types defined above. We say that a formula $\varphi(x, y)$ is algebraic in x if for every $\underline{a}$, $\varphi(x, \underline{a})$ is algebraic.

1.1. **Lemma.** *Assume* $A \subseteq Q$.

(1) $P_A^0 \subseteq P_A \subseteq P_A^* \subseteq P_A^\#$. *$P_A$ is ACL_A-closed in P_A^* and P_A^* is ACL_A-closed in* $P_A^\#$.

(2) *If $q|A$ is modular or orthogonal to p_a, $a \in Q$, then ACL_A on $P_A^\#$ and ACL_Q on $P_A^\#$ regarded as a subset of $P^\#$, agree.*

(3) *Assume $R \subseteq P_A^\#$, $r \in P^\#$ and $r \in \operatorname{ACL}_Q(R)$. Then for some $r' \in \operatorname{ACL}_A(R)$, r and r' are ACL_Q-interdependent.*

(4) *For countable A, P_A^* is countable.*

Proof. (1) The only less trivial inclusion is $P_A \subseteq P_A^*$. Let $r \in P_A$ and $r' = r|\operatorname{acl}(Q)$. Choose a finite $A' \subseteq A$ such that r' does not fork over A'. Since $r' \in \operatorname{ACL}(P^0)$, for some finite $B \subseteq Q$ containing A', r' does not fork over B and $\operatorname{Mlt}(r'|B)$ is finite. But q is stationary, hence $\operatorname{Mlt}(B/A')$ is finite, too. It follows that $\operatorname{Mlt}(r'|A')$ is finite.

(2) Suppose $X \subseteq P_A^\#$, $r \in P_A^\#$ and $r \in \operatorname{ACL}_Q(X)$. Revealing the definition we see that if q is orthogonal to p_a then obviously $r \in \operatorname{ACL}_A(X)$. Otherwise, $q|A$ is modular, and by Lemma 0.4, $r \in \operatorname{ACL}_Q(X)$ means that $r \in \operatorname{ACL}_A(X \cup \{q|A\})$. By Lemma 0.4, $q|A \in \operatorname{ACL}_A(X)$, and $r \in \operatorname{ACL}_A(X)$.

(3) W.l.o.g. A is finite, and we can assume r is based on Aa for some $a \in Q \setminus \operatorname{acl}(A)$. By assumption, there are $r_0, \dots, r_n \in R$, b_i realizing r_i and b realizing $r' = r|\operatorname{acl}(Aa)$ such that n is minimal possible. $\underline{b} = (b_0, \dots, b_n)$ is independent over A and

(a) $\underline{b} \perp a(A)$ and $b \in \operatorname{acl}(Aa\underline{b})$. If T is 1-based, then it is easy to prove that if c is a large enough fragment of $Cb(\underline{b}/Aab)$ then $r' = \operatorname{stp}(c/A)$ satisfies our demands (in fact, T is "locally" 1-based, at least close to types $r, r_0, \dots, r_n$).

This is the idea of the following proof. Let $\varphi(x, \underline{y}, \underline{a}')$ be a formula over $\operatorname{acl}(Aa)$ witnessing $b \in \operatorname{acl}(Aa\underline{b})$. So $D = \varphi(\mathfrak{C}, \underline{b}, \underline{a}')$ is finite and w.l.o.g. contained in $r'(\mathfrak{C})$. By the minimality of n we get

(b) $\dim_{Aa}(D) = 1$, that is b and D are interalgebraic over Aa. Let $q' = \operatorname{stp}(\underline{a}'/A)$ and $s = \operatorname{stp}(\underline{b}/A)$. We define an equivalence relation E on $s(\mathfrak{C})$ by $\underline{b}'E\underline{b}''$ iff for $\underline{a}''$ realizing $q'|A\underline{b}'\underline{b}''$, $\varphi(\mathfrak{C}, \underline{b}', \underline{a}'') = \varphi(\mathfrak{C}, \underline{b}'', \underline{a}'')$. Let $c = \underline{b}/E$. Hence $c \perp a(A)$. Let $r' = \operatorname{stp}(c/A)$. To finish it suffices to prove that b and c are interalgebraic over A. We have

(c) $b \in \operatorname{acl}(Aac)$. Indeed, suppose $f \in \operatorname{Aut}(\mathfrak{C})$, f fixes Aac pointwise, $f(b\underline{b}) = b'\underline{b}'$, and w.l.o.g. $b'\underline{b}' \perp b\underline{b}(Aac)$. Hence in particular

(d) $\underline{b}' \perp \underline{b}(Aac)$

(a) implies $\underline{b} \perp a(Ac)$, hence also $\underline{b}' \perp a(Ac)$, and by (d) we get $\underline{b}' \perp \underline{b}a(Ac)$ and $\underline{b}' \perp a(A\underline{b}c)$. Hence $a \perp \underline{b}\underline{b}'(Ac)$ and $a \perp c(A)$, which gives $a \perp \underline{b}\underline{b}'(A)$, and the more so $\underline{a}' \perp \underline{b}\underline{b}'(A)$, i.e. $\underline{a}'$ satisfies $q'|A\underline{b}\underline{b}'$. Since $\underline{b}/E = c = \underline{b}'/E$, we get $D = \varphi(\mathfrak{C}, \underline{b}', \underline{a}')$, hence $b' \in D$. This proves (c). Now we show

(e) $c \in \operatorname{acl}(AaD)$. Suppose $f \in \operatorname{Aut}(\mathfrak{C})$ fixes AaD pointwise, let $c'\underline{b}' = f(c\underline{b})$ and w.l.o.g. $c'\underline{b}' \perp c\underline{b}(AaD)$. In particular, $\underline{b}' \perp \underline{b}c(AAaD)$ and by (b) and (c), $D \subseteq \operatorname{acl}(Aac)$, hence $\underline{b}' \perp \underline{b}(Aac)$. This implies as above that $\underline{b}\underline{b}' \perp \underline{a}'(A)$, and $D = \varphi(\mathfrak{C}, \underline{b}, \underline{a}') = \varphi(\mathfrak{C}, \underline{b}', \underline{a}')$, because f is elementary. Hence $\underline{b}E\underline{b}'$, and $c' = c$.

(4) For a fixed finite $A' \subseteq A$, there are countably many strong types over A' with finitely many conjugates over A', because T is small. Hence P_A^* is countable.

Let ACL$'$ denote the ACL-dependence over $P_\varnothing^*$, that is $r \in \mathrm{ACL}'(R)$ iff $r \in \mathrm{ACL}(RP_\varnothing^*)$. We say that R is essentially ACL-closed in R' if $R \subseteq R'$ and for every $r \in \mathrm{ACL}(R) \cap R'$ there is an $r' \in R$ with $r \in \mathrm{ACL}(r')$. R is essentially ACL-closed if R is essentially ACL-closed in $P^\#$. Similarly we define the notion of an essentially ACL$'$-closed set. Hence Lemma 1.1(3) says that $P_A^\#$ is essentially ACL-closed. Every $r \in P_A^0$ is locally modular. Hence we can define F_p as the division ring corresponding to the pregeometry $(r(\mathfrak{C}), \operatorname{acl})$, and the choice of $a \in Q$ and $r \in P_A^0$ does not matter. By Lemma 1.1(2) we can omit A in ACL_A when $A \neq \varnothing$. By Lemma 0.5 and Proposition 0.6, $(P^\#, \mathrm{ACL})$ is projective over F_p. By Lemma 1.1(3), $P_A^\#$ is essentially a projective subspace of $P^\#$, hence is projective over F_p. Similarly, P_A and P_A^* are projective over F_p. By [Bu3] or [Ne1, 0.5], F_p is in fact a locally finite field. For $X, Y \subseteq P^\#$, $\mathrm{DIM}(X/Y)$ denotes the ACL-dimension of X over Y, and $\mathrm{DIM}(X)$ denotes the ACL-dimension of X. We define a notion of independence on $P^\#$. $X, Y \subseteq P^\#$ are independent over $Z \subseteq P^\#$ $(X \perp Y(Z))$ if for all finite $X' \subseteq X$, $Y' \subseteq Y$, $\mathrm{DIM}(X'/Z) = \mathrm{DIM}(X'/Y'Z)$. Clearly $X \perp Y(Z)$ implies $Y \perp X(Z)$ (see [Ne2]).

1.2. **Lemma.** *Assume* $A \subseteq Q$, $a \in Q \setminus \operatorname{acl}(A)$, $r \in P_a^0$. *Then* $r \notin \mathrm{ACL}(P_A^\#)$.

Proof. Suppose not. W.l.o.g. $A \neq \varnothing$. By Lemma 1.1, for some $s \in P_A^\#$, r and s are ACL_{Aa}-interdependent. That is for some b, c realizing $r|Q, s|Q$ respectively, b, c are acl-interdependent over Aa. In particular, $cA \perp a$ and $cA \not\perp b(a)$, showing r is not almost orthogonal to $\varnothing$. This contradicts Assumptions 0.1.

The next lemma and the following corollary show that ACL has a "local character", that is for $A, B, C \subseteq Q$ with $C \subseteq A \cap B$ and $A \perp B(C)$, $P^{\#}_A \perp P^{\#}_B(P^{\#}_C)$.

1.3. **Lemma.** *Assume* $A, B, C \subseteq Q$, $A \perp B(C)$, $r \in P^{\#}_A$, $r' \in P^{\#}_B$, $r \in \mathrm{ACL}(r')$. *Then for some* $r'' \in P^{\#}_C$, $r \in \mathrm{ACL}(r'')$.

Proof. W.l.o.g. $A = \{a\} \cup C$, $B = \{b\} \cup C$, $a, b \in Q \setminus \mathrm{acl}(C)$. For simplicity assume $C = \varnothing$. So we have $a \perp b$, $r_a \in P^{\#}_a$, $r_b \in P^{\#}_b$ and $r_a \in \mathrm{ACL}(r_b)$. We want to find an $r \in P^{\#}_{\varnothing}$ such that $r_a \in \mathrm{ACL}(r)$. Let a' realize $r_a|ab$, b' realize $r_b|ab$ and $b' \in \mathrm{acl}(aa'b)$. Let c be a sufficiently large finite fragment of $Cb(bb'/aa')$ such that $Cb(bb'/aa') \subseteq \mathrm{acl}(c)$. Hence $c \in \mathrm{acl}(aa')$, and since $b \perp aa'$, $c \perp b$. We have $U(c/a) = U(a'/a)$ and $a' \in \mathrm{acl}(ca)$, hence it suffices to prove

(a) $c \perp a$. Then $r = \mathrm{stp}(c/\varnothing)$ will satisfy our demands. To prove (a) it suffices to show $c \subseteq \mathrm{acl}(bb')$. This follows because T is "locally" 1-based. We can prove $c \subseteq \mathrm{acl}(bb')$ directly. Let $b_0b'_0 = bb'$, $b_1b'_1$, $b_2b'_2, \ldots$ be a Morley sequence in $\mathrm{stp}(bb'/aa')$. Let $I = \{b_0, b_1, \ldots\}$, $J = \{b'_0, b'_1, \ldots\}$. $b \perp aa'$ implies $I \perp aa'$, hence $c \perp I$, $b' \in \mathrm{acl}(aa'b)$ implies $J \subseteq \mathrm{acl}(Iaa')$, hence $\dim(J/I) \leq 2$ and $c \in \mathrm{acl}(IJ)$. $\dim(J/I) \leq 1$ would imply $U(c/\varnothing) = 1$, and $c \subseteq \mathrm{acl}(bb')$. So we can assume $\dim(J/I) = 2$. This will lead to a contradiction.

If q is orthogonal to p_a, then $c \in \mathrm{acl}(IJ)$ implies $c \perp a$, and we are done. So we can assume q is nonorthogonal to p_a, and then q is locally modular. We have J is pairwise I-independent, but not I-independent. We shall transfer this situation to $r_b(\mathfrak{C})$, to reach a contradiction with modularity.

Let $r_i = \mathrm{stp}(b'_i/b_i)$. We have $b'_i \perp I(b_i)$. By transitivity of ACL, $r_b \in \mathrm{ACL}(r_i)$, hence by Lemma 0.4, there is b''_i realizing $r_b|I$ such that $b''_i \in \mathrm{acl}(b'_iI)$, and $b''_0 = b'$. Let $J'' = \{b''_0, b''_1, \ldots\}$. Hence $\dim(J''/I) = 2$ and for $i \neq j$, $b''_i \perp b''_j(I)$. Let b^* realize $r_b|IJ$. So $r_b|bb^*$ is modular, and nonorthogonal to $q|b$, which is also modular. Thus we can choose $I^* = \{b^*_1, b^*_2, \ldots\}$, a Morley sequence in $r' = r_b|bb^*$, such that b^*_i is interalgebraic over bb^* with b_i. Add bb^* to the signature. So we have $\dim(J''/I^*) = 2$, and for $i \neq j$, $b''_i \perp b''_j(I^*)$. We can assume that for $i, j > 2$,

(b) $b^*_ib''_i \equiv b^*_jb''_j(b^*_1b''_1b^*_2b''_2)$. We have $b''_3 \in \mathrm{acl}(b''_1b''_2I^*)$, hence by modularity of r', there is a' realizing r' with $a' \in \mathrm{acl}(b^*_1b''_1b^*_2b''_2) \cap \mathrm{acl}(b''_3b^*_3)$. By (b), $a' \in \mathrm{acl}(b''_4b^*_4)$ as well. It follows $b''_3 \not\perp b''_4(b^*_3b^*_4)$, hence $b''_3 \not\perp b''_4(I^*)$, a contradiction.

1.4 **Corollary.** *Assume* $A, B \subseteq Q$, $C \subseteq A \cap B$ *and* $A \perp B(C)$. *Then* $P^{\#}_A \perp P^{\#}_B(P^{\#}_C)$, $P^*_A \perp P^*_B(P^*_C)$ *and* $P_A \perp P_B(P_C)$.

Proof. For example we prove $P^{\#}_A \perp P^{\#}_B(P^{\#}_C)$. Suppose not. Then there are finite $X \subseteq P^{\#}_A$, $Y \subseteq P^{\#}_B$ with $\mathrm{DIM}(X/P^{\#}_CY) < \mathrm{DIM}(X/P^{\#}_C)$. By modularity of ACL, there is an $r^* \in \mathrm{ACL}(P^{\#}_A) \cap \mathrm{ACL}(P^{\#}_B) \setminus \mathrm{ACL}(P^{\#}_C)$. By Lemma 1.1(3) there are $r \in P^{\#}_A$ and $r' \in P^{\#}_B$, ACL-interdependent with r^*. By Lemma 1.3, $r^* \in \mathrm{ACL}(P^{\#}_C)$, a contradiction.

In the next proposition we see how to reduce (P) to the problem of counting the isomorphism types of ACL-closed countably dimensional subsets of P^*.

1.5. **Proposition.** *Assume M is a countable model of T and $A = q(M)$.*

(1) *Let $R(M) = \{r \in P_A\colon r$ is realized in $M\}$. Then $R(M)$ is* ACL-*closed in P_A.*

(2) *Conversely, if $R \subseteq P_A^*$ is* ACL-*closed in P_A^* then there is N with $A = q(N)$ and $R = \{r \in P_A^*\colon r$ is realized in $N\}$.*

Proof. $R(M)$ is ACL-closed in P_A by Lemma 0.4. If $R \subseteq P_A^*$ is ACL-closed in P_A^*, then by Lemma 1.1(4) we can find the desired N by the omitting types theorem.

1.6. **Corollary.** *To compute $I(p, \aleph_0)$ it suffices to determine the isomorphism types of* ACL-*closed in P_A subsets of P_A, where $A = q(M)$ for a countable M.*

Proof. We have $p(M) = \bigcup\{p_a(M)\colon a \in A = q(M)\} \subseteq \bigcup\{r(M)\colon r \in R(M)\}$, where $R(M) = \{r \in P_A\colon r$ is realized in $M\}$. By local modularity, any two types in $R(M)$ have the same dimension n_M in M. Hence the isomorphism type of $p(M)$ is determined by the dimension of A, n_M and the isomorphism type of $R(M)$.

Consequently our further study concentrates on ACL-closed subsets of P^*. In fact it suffices to investigate only ACL-closed subsets of P_A, but it turns out that it is not easier at all. By Proposition 1.6, any two nonisomorphic ACL-closed countably dimensional subsets of P^* correspond to nonisomorphic countable models of T. In particular, $I(T, \aleph_0) < 2^{\aleph_0}$ implies there are $< 2^{\aleph_0}$ isomorphism types of ACL-closed countably dimensional subsets of P^*. Notice that we need not to bother about subsets of P_A^* with A finite dimensional. The following lemma is a reformulation of [Bu2, 5.2(a)] or [Bu4, 1.14].

1.7. **Lemma.** *If $A \subseteq Q$ has finite dimension then P_A^* has finite* ACL-*dimension.*

1.8. **Corolllary.** *If $A \subseteq Q$ is finite then there are only countably many nonisomorphic* ACL-*closed subsets of* ACL(P_A^*). *In particular, there are countably many isomorphism types in the set $\{p(M)\colon q(M)$ has finite dimension$\}$.*

Proof. Follows by Lemmas 1.1, 1.7 and Corollary 1.6.

The following lemma, proved in [Bu4, §2], is fundamental in this paper.

1.9. **Lemma.** *Q is locally modular.*

An easy argument from [Bu4, §1] gives also

1.10. **Assumption.** Q is nontrivial.

By Lemma 1.9 we can define F_q as the division ring corresponding to the pregeometry (Q, acl). Now we have two pregeometries: (Q, acl) and (P^*, ACL) (or (P, ACL) if you like). Counting the isomorphism types of ACL-closed subsets of P^* has mostly a geometrical character, and utilizes the interaction between (Q, acl) and (P^*, ACL). There were two breakthroughs in the understanding of properly weakly minimal sets. The first was the Buechler's discovery that such sets are locally modular [Bu1]. This introduced geometry into the subject. The second was Hrushovski's result, connecting with a modular regular type a connected modular regular group G such that the forking dependence on the generic type of G may be regarded just as a vector space dependence [H]. This introduced more algebra and stable groups into the subject. We do a similar thing in this paper, explaining the structure of U-rank 2 types. Now we are at the geometrical stage. Geometrical means are quite restricted, using them we shall be able to prove for example that F_q being finite implies that F_p is finite. Then in §§2 and 3 we shall translate the ACL′-dependence on P^* into a linear dependence in some vector space over F_p. This will solve the problem of counting $I(p|E, \aleph_0)$ for some finite set of parameters E. Note however that in the Hrushovski result in [H] mentioned above we also need some parameters.

Later we shall be extending T by adding new constants. Now we shall discuss this procedure. By Corollary 1.8 we may restrict ourselves to the countable models with $\dim(q(M))$ infinite. Hence we can quite harmlessly add any finite independent subset A of Q to the signature. Then we replace p and q by $p|A$, $q|A$, and for $a \in Q \setminus \mathrm{acl}(A)$, p_a by $p_a|Aa$ (by Assumptions 0.1, p_a is weakly orthogonal to $\mathrm{tp}(A/a)$). Hence the new P^0 and P somewhat change, but P^* remains essentially the same, and if we classify the countably dimensional ACL-closed subsets of P^*, then this will give also a classification of the sets $p(M)$ for the old p. It is more convenient to work with ACL′ rather than with ACL. This is because it may happen that for $A, B \subseteq Q$ with $A \perp B$, $P_A^\# \not\perp P_B^\#$, while by Corollary 1.4 we have then $P_A^\# \perp P_B^\#(P_\varnothing^\#)$. That is, it is really ACL′, and not ACL, which has "local character'.

In Theorem 2.2 we shall give a reduction of the problem of counting ACL′-closed subsets of P^* to a problem from linear algebra. Now if there are $2^{\aleph_0}$ isomorphism types of countably dimensional ACL′-closed subsets of P^*, then this gives clearly also $I(T, \aleph_0) = 2^{\aleph_0}$.

Suppose there are countably many nonisomorphic countably dimensional ACL′-closed subsets of P^*. We would like to count countably dimensional ACL-closed subsets of P^*. Here we do not have a general method, and must deal separately with each particular case. In §4 we shall see examples of such a reasoning. From now on we fix the following assumption.

1.11. **Assumption.** Q is modular.

The local character of ACL enables us to define the following concept of a basis $\pi(r)$ for $r \in P^*$.

1.12. **Definition.** Let $r \in P^*$. Define $\pi(r)$ as the minimal set A such that $A = \mathrm{acl}(A) \cap Q$ and $r \in \mathrm{ACL}(P_A^*)$. Let $n(r) = \dim(\pi(r))$. For $R \subseteq P^*$ let $\pi(R) = \mathrm{acl}(\bigcup\{\pi(r) : r \in R\}) \cap Q$. For $r \in P$ let $n'(r)$ be the minimal n such that for some $A \subseteq Q$ of size n, $r \in \mathrm{ACL}(P_A^0)$.

Notice that by Corollary 1.4 and Assumption 1.11, the definition of $\pi(r)$ is correct, because if $r \in \mathrm{ACL}(P_A^*) \cap \mathrm{ACL}(P_B^*)$ and $C = \mathrm{acl}(A) \cap \mathrm{acl}(B) \cap Q$ then $A \perp B(C)$ and $r \in \mathrm{ACL}(P_C^*)$. In the next lemma we define important coefficients n_a and n_b.

1.13. **Lemma.** (1) $n_a = \max\{n(r) : r \in P^*\}$ *is finite. Hence also* $n_b = \max\{n(r) : r \in P\}$ *is finite.*

(2) $n_c = \max\{n'(r) : r \in P\}$ *is finite.*

Proof. We shall prove only (1) as (2) is similar. First we prove the following triangle inequality.

(Δ) If $r, r', r'' \in P^*$ are pairwise ACL-independent and $r'' \in \mathrm{ACL}(r, r')$, then $|n(r) - n(r')| \leq n(r'') \leq n(r) + n(r')$.

The right-hand side inequality follows by the definition of $n(r)$. Then the left-hand side inequality follows by the exchange principle.

Now suppose that there are r with $n(r)$ arbitrarily large. Then we can choose $\{A_i: i < \omega\}$, an independent family of subsets of Q, and $r_i \in P^*_{A_i}$ such that $n(r_i) = \dim(A_i)$ and if $k_i = n(r_i)$, then for every $i > 0$,

$$(*) \qquad k_0 + \cdots + k_{i-1} < k_i.$$

Let $R = \{r_i: i < \omega\}$, $R_{<i} = \{r_j: j < i\}$, $R_{>i} = \{r_j: j > i\}$ and $R_{\neq i} = \{r_j: j \neq i\}$.

1.14. **Claim.** (1) If $r \in \mathrm{ACL}(R_{<i})$ then $n(r) \leq k_0 + \cdots + k_{i-1}$.
(2) If $r \in \mathrm{ACL}(R_{>i})$ then $n(r) > k_0 + \cdots + k_i$.
(3) If $r \in \mathrm{ACL}(R_{\neq i})$ then $n(r) \neq k_i$.

Proof. (1) follows immediately by (Δ). In (2), choose a minimal $R' \subseteq R_{>i}$ such that $r \in \mathrm{ACL}(R')$. Choose $r' \in R'$ with maximal $n(r')$. Then by the exchange principle we get $n(r') \leq n(r) + \sum\{n(r''): r'' \in R' \setminus \{r'\}\}$. Hence by $(*)$ we are done.

(3) Let $r \in \mathrm{ACL}(R_{\neq i})$. Hence there are $r' \in \mathrm{ACL}(R_{<i})$ and $r'' \in \mathrm{ACL}(R_{>i})$ with $r \in \mathrm{ACL}(r', r'')$. If $n(r) = k_i$ then by (1) and (Δ) we get $n(r'') \leq k_1 + \cdots + k_i$, contradicting (2) and $(*)$.

For $X \subseteq \omega$ let $R_X = \{r_i: i \in X\}$. By the claim we see that $i \in X$ iff for some $r \in \mathrm{ACL}(R_X)$, $n(r) = k_i$. Hence for distinct X and X', $\mathrm{ACL}(R_X)$ and $\mathrm{ACL}(R_{X'})$ are nonisomorphic, contradicting $I(T, \aleph_0) < 2^{\aleph_0}$.

In the course of proving (P) or Vaught's conjecture we usually proceed through a series of dichotomies. Each such dichotomy consists in discerning a property A which implies that T has $2^{\aleph_0}$ countable models ("a many-model argument"). Then theories fall into two categories, depending on whether they have property A or not. And as theories satisfying A satisfy also Vaught's conjecture, we can deal further on only with theories which do not have property A. In our case many model arguments will usually consist in encoding sets of natural numbers into models of T, and a number will be encoded as a dimension of some object within M. These objects will often be subsets of P^*. The many-model arguments in this paper are closely related to ACL. Lemma 1.13 is an example of a many-model argument. Usually such an argument relies on the local character of ACL (Corollary 1.4). As an application of Lemma 1.13 we establish a firm relationship between P^0_A and P_A.

1.15. **Proposition.** *Assume $A \subseteq Q$ and $\dim(A) \geq n_c$. Then $P_A \subseteq \mathrm{ACL}(P^0_A)$.*

Proof. W.l.o.g. $A = \mathrm{acl}(A) \cap Q$. Let $r \in P_A$. Choose $B = \mathrm{acl}(B) \cap Q$ with $\dim(B) \leq n_c$ and $r' \in \mathrm{ACL}_B(P^0_B)$ such that $r \in \mathrm{ACL}(r')$. So r' is based on B. Let $C = A \cap B$. We have $A \perp B(C)$, hence by Corollary 1.4 there is $r'' \in P_C$ ACL-interdependent with both r and r'. Within A we can find a copy B^* of B over C. Let r^* be a conjugate of r' over B^*. Hence $r^* \in \mathrm{ACL}(r'')$, $r^* \in \mathrm{ACL}_A(P^0_A)$. Consequently, $r \in \mathrm{ACL}(r^*)$ and $r \in \mathrm{ACL}(P^0_{B^*}) \subseteq \mathrm{ACL}(P^0_A)$.

We will use the following theorem, which is a local version of NOTOP, proved in [Bu4, §2] (see also [Ne3, 3.1] for a weaker version and [Bu5, 3.3] for a stronger version in the unidimensional case).

1.16. **Theorem.** *Assume $A, B \subseteq Q$. Then $P^*_{AB} \subseteq \mathrm{ACL}(P^*_A \cup P^*_B)$ and $P_{AB} \subseteq \mathrm{ACL}(P_A \cup P_B)$.*

1.17. **Corollary.** *Let $A \subseteq B \subseteq Q$ and $\{b_0, \dots, b_n\}$ be a basis of B over A. Then $P^*_B \subseteq \mathrm{ACL}(P^*_A \cup \bigcup_{i \leq n} P^*_{b_i})$ and $P_B \subseteq \mathrm{ACL}(P_A \cup \bigcup_{i \leq n} P_{b_i})$.*

Proof. Apply Theorem 1.16 consecutively.

Now we define some more coefficients.

1.18. **Definition.** Let $n_d = \mathrm{DIM}(P_a^*/P_\varnothing^*)$ and $n_e = \mathrm{DIM}(P_a/P_\varnothing^*)$ for any $a \in Q$. By Lemma 1.2, $1 \le n_e \le n_d$, and by Lemma 1.7, n_e, n_d are finite. Notice also that since $P_a \perp P_\varnothing^*(P_\varnothing)$, $n_e = \mathrm{DIM}(P_a/P_\varnothing)$. Hence one might try to prove (P) by induction with respect to n_e or n_d. Any conceivable classification of ACL-closed subsets of P^* should consist in a decomposition of an ACL-closed subset of P^* into small pieces. This leads to the notion of a free decomposition introduced below.

1.19. **Definition.** Assume $R \subseteq P^*$ is ACL$'$-closed. We say that $R' \subseteq P^*$ is a strong subset of R $(R' < R)$ if $R' = \mathrm{ACL}'(R \cap P^*_{\pi(R')})$. We say that $\{R_i, i \in I\}$ is a free decomposition of R if

(1) R_i is ACL$'$-closed, $R_i < R$ and $\mathrm{DIM}(R_i/P_\varnothing^*) > 0$,

(2) $\{\pi(R_i), i \in I\}$ is independent and

(3) $R = \mathrm{ACL}'(\bigcup_i R_i)$.

We say that R is decomposable if there is a free decomposition $\{R_i, i \in I\}$ of R with $|I| \ge 2$. Otherwise we say that R is indecomposable.

1.20. *Remark.* (1) Assume $R \subseteq P^*$ is ACL$'$-closed and $\{R_i, i \in I\}$ is a free decomposition of R. Then $\dim(\pi(R)) = \sum_{i\in I} \dim(\pi(R_i))$, $\mathrm{DIM}(R/P_\varnothing^*) = \sum_{i\in I} \mathrm{DIM}(R_i/P_\varnothing^*)$ and $\mathrm{DIM}(P^*_{\pi(R)}/R) = \sum_{i\in I} \mathrm{DIM}(P^*_{\pi(R_i)}/R_i)$.

(2) Assume $A = \{a_i, i \in I\} \subseteq Q$ is independent. Then $\{\mathrm{ACL}'(P^*_{a_i}), i \in I\}$ is a free decomposition of $\mathrm{ACL}'(P_A^*)$.

Proof. Follows by Corollary 1.4 and Theorem 1.16.

1.21. **Proposition.** (1) *If $n_d = 1$ then every* ACL$'$*-closed $R \subseteq P^*$ is of the form* $\mathrm{ACL}(P_A^*)$, *where $A = \pi(R)$.*

(2) *If $n_e = 1$ then for every $R \subseteq P$,* $\mathrm{ACL}'(R)$ *is of the form* $\mathrm{ACL}'(P_A)$, *where $A = \pi(R)$.*

Proof. Trivial.

Let $\{r_0, \dots, r_n\}$ be a basis of $P_\varnothing^*$, let c_i realize r_i, and let $C = \{c_0, \dots, c_n\}$. If we add C to the signature, then every ACL-closed $R \subseteq P^*$ becomes ACL$'$-closed, because all types from the old $P_\varnothing^*$ become modular now, hence the new $P_\varnothing^* = \varnothing$. In view of Corollary 1.6 and Proposition 1.21 we get

1.22. **Corollary.** *If $n_d = 1$ or $n_e = 1$ and $p' = p|C$ then $I(p', \aleph_0) = \aleph_0$.*

The main drawback of this corollary is that C is not embeddable in every model of T. Hence we cannot conclude immediately that $I(p, \aleph_0) = \aleph_0$ in this case. We shall deal with this problem in §4. Using the current geometrical set-up, we have a rather limited understanding of the structure of an ACL-closed subset of P^*. Thus we are able to produce a many-model argument only in extreme cases, for example when F_p and F_q differ very much. The next theorem will not be used anywhere later, and also in §3 we shall prove it by other means. We include it here as an example of what we can prove using the current geometrical set-up.

1.23. **Theorem.** *If $n_d > 1$ and F_q is finite then F_p is finite. Supposing F_q is finite and F_p is infinite, we will construct $2^{\aleph_0}$ nonisomorphic* ACL*-closed subsets of P^* with countable dimension.*

1.24. **Definition.** We say that $R \subseteq P^*$ is almost full if R is ACL'-closed and for $A = \pi(R)$, $\mathrm{DIM}(P_A^*/R) = 1$ and there is no $a \in A$ with $P_a^* \subseteq R$.

1.25. **Lemma.** *Assume there are almost full* R_i, $0 < i < \omega$, *with* $\{\pi(R_i), 0 < i < \omega\}$ *independent and* $\dim(\pi(R_i)) = i$. *Then* $I(T, \aleph_0) = 2^{\aleph_0}$.

Proof. Let $A_i = \pi(R_i)$. For $X \subseteq \omega \backslash \{0\}$ let $R_X = \mathrm{ACL}(\bigcup_{i \in X} R_i)$, $A_X = \mathrm{acl}(\bigcup_{i \in X} A_i) \cap Q$ (hence $A_X = \pi(R_X)$), $X_{>i} = \{j \in X : j > i\}$, $X_{\neq i} = \{j \in X : j \neq i\}$. We prove that for $X \neq Y$, R_X and R_Y are nonisomorphic. Fix an $X \subseteq \omega$. By Corollary 1.4, $\{P_{A_i}^*, i < \omega\}$ is independent (over $P_\varnothing^*$). Hence for $i \in X$, $R_i < R_X$. We prove

(a) Suppose $R < R_X$ is almost full, $A = \pi(R)$, $i \in X$ and $A \not\perp A_i(A')$, where $A' = A_{X_{\neq i}}$. Then $A \perp A'$.

Indeed, suppose $a \in A \cap A'$ and $b \in A \backslash A'$. There are $c \in A_i$ and $d \in A'$ with $b \in \mathrm{acl}(c, d)$ and $c \in \mathrm{acl}(b, d)$, we have $P_A^* \subseteq \mathrm{ACL}(RP_A^*)$ (because $P_a^* \subseteq R$ and $\mathrm{DIM}(P_A^*/R) = 1$). Hence $P_b^* \subseteq \mathrm{ACL}(R_X P_{A'}^*) = \mathrm{ACL}(R_{X_i} P_{A'}^*)$ (since $P_{A_i}^* \perp P_{A'}^*(P_\varnothing^*)$). Also, $c \in \mathrm{acl}(b, d)$ implies $P_c^* \subseteq \mathrm{ACL}(R_{X_i} P_{A'}^*)$, and $P_{A_i} \subseteq \mathrm{ACL}(P_c^* R_X)$. Together we get $P_{A_i}^* \subseteq \mathrm{ACL}(R_{X_i} P_{A''}^*)$. Since $P_{A_i}^* \perp P_{A'}^*(P_\varnothing^*)$, $P_{A_i}^* \subseteq R_i$, a contradiction. To finish, we prove

(b) $i \notin X$ iff for every almost full $R < R_X$ maximal in R_X, if $\dim(\pi(R)) = i$ then there are almost full $R^1, \dots, R^n < R_X$ (for some n) with $\dim(\pi(R^j)) > i$ and $R \subseteq \mathrm{ACL}(R^1 \cup \cdots \cup R^n)$.

$\rightarrow$. Suppose $R < R_X$ is almost full and maximal in R_X with respect to this condition, and $\dim(\pi(R)) = i$. Let $A = \pi(R)$ and suppose $i \notin X$, $A_X = \pi(R_X)$, hence $A \subseteq A_X$. By induction on $j < i$ we prove that $A \subseteq A_{X_{>j}}$. Suppose $A \subseteq A_{X_{>(j-1)}}$. If $j \notin X$, we are done. Otherwise, by (a), $A \perp A_j(A_{X_{>j}})$ (because $\dim(A_j) < \dim(A)$), hence $A \subseteq A_{X_{>j}}$. So $A \subseteq A_{X_{>(i-1)}} = A_{X_{>i}}$, since $i \notin X$. It follows that $R \subseteq R_{X_{>i}}$.

$\leftarrow$. Suppose $i \in X$. We have proved above that for any almost full $R' < R_X$ with $\dim(\pi(R')) > i$, $\pi(R') \subseteq A_{X_{>i}}$. It follows that R_i is maximal almost full in R_X, and there are no almost full $R^1, \dots, R^n < R_X$ with $\dim(\pi(R^j)) > i$ and $R_i \subseteq \mathrm{ACL}(R^1 \cup \cdots \cup R^n)$.

Proof of Theorem 1.23. By Lemma 1.25 it suffices to find for every $i < \omega$ an almost full $R \subseteq P^*$ with $i = \dim(\pi(R))$. We find such an R by induction on i. It is easy to find an almost full R with $\dim(\pi(R)) = 1$. So suppose R is almost full, $A = \pi(R)$, $\dim(A) = i > 0$ and we must find an almost full R' with $\dim(\pi(R')) = i + 1$. Let $a \in Q \backslash A$. Choose an ACL'-closed $R_0 \subseteq \mathrm{ACL}(P_a^*)$ with $\mathrm{DIM}(P_a^*/R_0) = 1$. Let $A' = \mathrm{acl}(Aa) \cap Q$. We see that $\mathrm{DIM}(P_{A'}^*/RR_0) = 2$. Since F_p is infinite, there are $r_n \in P_{A'}^* \backslash \mathrm{ACL}(RR_0)$, $n < \omega$, such that

(c) $r_n \notin \mathrm{ACL}(RR_0 r_m)$ for $n \neq m < \omega$. On the other hand, as F_q is finite, there are finitely many $a_0, \dots, a_k \in A'$ such that for every $b \in A'$, $b \in \mathrm{acl}(a_i)$ for some $i \leq k$. Of course for every i, $P_{a_i}^* \not\subseteq \mathrm{ACL}(RR_0)$. By (c) for every $i \leq k$, there is at most one n such that $P_{a_i}^* \subseteq \mathrm{ACL}(RR_0 r_n)$. Hence for some n', there is no $i \leq k$ with $P_{a_i} \subseteq \mathrm{ACL}(RR_0 r_{n'})$. Let $R' = \mathrm{ACL}(RR_0 r_{n'})$. We see that R' is almost full and $\pi(R') = A'$, $\dim(A') = i + 1$.

2. The main result

We shall translate ACL$'$-dependence on P^* into a linear dependence in some vector space. This is possible if we fix the following assumptions. We can fix them by [H] (also see the discussion in §1).

2.1. **Assumptions.** G is a type definable over $\varnothing$ weakly minimal connected group in $\mathfrak{C}$, q is the generic type of G, $Q = q(\mathfrak{C})$ is modular, F_q is the division ring of definable almost over $\varnothing$ pseudo-endomorphisms of G.

Let $\mathscr{A} = \mathrm{acl}(\varnothing) \cap G$. By [H], $G/\mathscr{A}$ is a vector space over F_q and acl-dependence on G translates just into F_q-linear dependence on $G/\mathscr{A}$. We fix the following notation. For $a \in G$ let $a_{\mathscr{A}}$ be $a + \mathscr{A}$, an element of $G/\mathscr{A}$. For $A \subseteq G$, $A_{\mathscr{A}}$ denotes $\{a_{\mathscr{A}} : a \in A\}$. Let $G_{\mathscr{A}}$ be the set $G/\mathscr{A}$ with the structure of right vector space over F_q. Suppose K is a division ring, V is a right vector space over K and $L \subseteq M_{n\times n}(K)$ is a division subring of $M_{n\times n}(K)$. Then we can regard V^n as a right vector space over L: L acts on V^n by matrix multiplication on the right. For $\underline{v} = (v_1, \dots, v_n) \in V^n$ let $\pi'(\underline{v}) = K\text{-}\mathrm{span}(v_1, \dots, v_n) = v_1K + \cdots + v_nK \subseteq V$. For $W \subseteq V^n$, $\pi'(W)$ is the K-subspace of V generated by $\pi(\underline{v})$, $\underline{v} \in W$.

Assume W, W' are L-subspaces of V^n. We say that W, W' are isomorphic in V (or just : isomorphic), if there is a K-linear isomorphism $f: \pi(W) \to \pi(W')$ such that f induces an L-linear isomorphism of W and W'. Now let Γ be a group of automorphisms of the division ring K such that every $\gamma \in \Gamma$ preserves L, that is $\gamma[L] = L$. We say that W, W' are Γ-isomorphic in V if for some $\gamma \in \Gamma$ there is a bijection f of $\pi'(W)$ and $\pi'(W')$ (called a γ-isomorphism), which is a group isomorphism such that for every $a \in \pi'(W)$ and $\alpha \in K$, $f(a\alpha) = f(a)\gamma(\alpha)$, and $f[W] = W'$. Notice that if W, W' are isomorphic then W, W' are Γ-isomorphic (with $\gamma = \mathrm{id}$).

Let Γ_q be the group of automorphisms of F_q induced by $\mathrm{Aut}(\mathfrak{C})$. We will use $\mathbf{0}$ to denote a matrix or tuple of suitable size consisting of zeros only, I will denote the identity matrix of a suitable size. Now we shall formulate the main results of the paper. They will be proved in this and the next section.

2.2. **Theorem.** *There is an embedding $i: F_p \to i[F_p] = F \subseteq M_{n_a\times n_a}(F_q)$ which is a ring monomorphism. Moreover, after adding a finite subset of* $\mathrm{acl}(\varnothing)$ *to the signature, there is a correspondence (that is a binary relation) Φ between P^* and G^{n_a} such that the following hold.*

(1) *Φ and F are invariant under automorphisms of $\mathfrak{C}$.*

(2) $\mathrm{Dom}(\Phi) = P^*$ *and* $G^{n_a} \setminus \mathrm{acl}(\varnothing) \subseteq \mathrm{Rng}(\Phi) \subseteq G^{n_a}$.

(3) *For $r^1, \dots, r^n \in P^*$ and $\underline{a}^1, \dots, \underline{a}^n \in G^{n_a}$ with $\Phi(r^i, \underline{a}^i)$, $r^1, \dots, r^n$ are* ACL$'$*-independent iff $\underline{a}^1_{\mathscr{A}}, \dots, \underline{a}^n_{\mathscr{A}}$ are linearly independent over F.*

(4) *For $r \in P^*_{\varnothing}$, $\Phi(r, 0)$ holds.*

(5) *If r is a stationarization of p_a, $a \in Q$, then $\Phi(r, (a, \mathbf{0}))$ holds.*

2.3. **Corollary.** *Let i, Φ be as in Theorem 2.2. Φ induces a bijection Ψ between* ACL$'$*-closed subsets of P^* and F-subspaces of $G^{n_a}_{\mathscr{A}}$. Assume $R, R' \subseteq P^*$ are* ACL$'$*-closed. Then R, R' are isomorphic iff $\Psi(R)$, $\Psi(R')$ are Γ_q-isomorphic in $G_{\mathscr{A}}$.*

Proof. For ACL$'$-closed $R \subseteq P^*$ let $\Psi(R)$ be the set $\{\underline{a}_{\mathscr{A}} \in G^{n_a}_{\mathscr{A}} : \underline{a}_{\mathscr{A}} = \mathbf{0}$ or for some $r \in R$, $\Phi(r, \underline{a})$ holds$\}$. By Theorem 2.2, $\Psi(R)$ is an F-subspace of $G^{n_a}_{\mathscr{A}}$. By Theorem 2.2(1), (3), for $r \in P^*$ and $\underline{a} \in G^{n_a}$ with $\Phi(r, \underline{a})$, $\underline{a}$ and $\pi(r)$ are interalgebraic, hence if $A(R) = \pi(R) \cup \{0\}$, then $\pi'(\Psi(R)) = A(R)_{\mathscr{A}}$. It is clear that Ψ is a bijection between ACL$'$-closed subsets of P^* and F-subspaces of $G^{n_a}_{\mathscr{A}}$. Also by Theorem 2.2, $\gamma[F] = F$ for every $\gamma \in \Gamma_q$. Now suppose R, R' are ACL$'$-closed and isomorphic subsets of P^*. That is, for some $f \in \mathrm{Aut}(\mathfrak{C})$, $f[R] = R'$. By Theorem 2.2(1), f induces a γ-isomorphism of $\Psi(R)$ and $\Psi(R')$ (with $\gamma \in \Gamma_q$ induced by f).

Conversely, suppose $W = \Psi(R)$, $W' = \Psi(R') \subseteq G^{n_a}_{\mathscr{A}}$ are γ-isomorphic for some $\gamma \in \Gamma_q$. Hence there is a γ-isomorphism f_0 of $\pi'(W)$ and $\pi'(W')$ with $f_0[W] = W'$. By compactness it is easy to see that there is an $f \in \mathrm{Aut}(\mathfrak{C})$ with $f|F_q = \gamma$ such that $f_0(a_{\mathscr{A}}) = f(a)_{\mathscr{A}}$ for $a \in \pi(R)$. By Theorem 2.2, $f[R] = R'$.

If we want to consider only P (which is sufficient to determine the value of $I(p, \aleph_0)$), we get the following versions of Theorem 2.2 and Corollary 2.3.

2.2$'$. **Theorem.** *There is an embedding $i' \colon F_p \to i'[F_p] = F' \subseteq M_{n_b \times n_b}[F_q]$ which is a ring monomorphism. Moreover, after adding a finite subset of* $\mathrm{acl}(\varnothing)$ *to the signature, there is a correspondence Φ' between P and G^{n_b} such that conditions* (1)–(5) *from Theorem* 2.2 *hold with Φ replaced by Φ', P^* by P and n_a by n_b.*

2.3$'$. **Corollary.** *Let i', Φ' be as in Theorem 2.2$'$. Φ' induces a bijection Ψ' between* ACL$'$*-closed subsets of* ACL$'(P)$ *and F'-subspaces of $G^{n_b}_{\mathscr{A}}$. If $R, R' \subseteq$* ACL$'(P)$ *are* ACL$'$*-closed then R, R' are isomorphic iff $\Psi'(R)$ and $\Psi'(R')$ are Γ_q-isomorphic in $G_{\mathscr{A}}$.*

This and the next section is devoted to the proof of Theorem 2.2. After the proof of this theorem we shall indicated how to prove Theorem 2.2$'$. Then we shall discuss some further implications of Theorems 2.2 and 2.2$'$. The key to the proof of Theorem 2.2 is an analysis of ACL$'$ on P^*. Notice that if $X \subseteq P^*$ is essentially ACL$'$-closed then ACL$'$ is modular on X and X with ACL$'$-dependence is projective over F_p. It would be nice if just P^0 were essentially ACL$'$-closed, then we would not have to bother about "imaginary" types in P^*. Our current goal is to choose a nice essentially ACL$'$-closed $X \subseteq P^*$ with ACL$'(X) = P^*$. The point is that for every $r \in P^*$, out of the many types $r' \in P^*$ ACL$'$-interdependent with r, we need to include into X only one of a specific "sort". This will make the ACL$'$-dependence on X easier to understand. We shall eventually find finitely many types $r_1, \dots, r_k \in P^*$ (for some k) such that the set X of copies of $r_1, \dots, r_k$ will be essentially ACL$'$-closed and ACL$'(X) = P^*$.

We cannot expect that for every $r \in P^*_A$ with $A = \pi(r)$, and b realizing r, $\mathrm{tp}(b/A)$ is stationary. To overcome this difficulty in the process of proving Theorem 2.2 we will add to the signature a finite subset of $\mathrm{acl}(\varnothing)$. We will use the following trivial fact.

2.4. **Fact.** Assume $\mathrm{tp}(ab)$ is stationary. Then for every $A \perp b$, $\mathrm{tp}(a/b)$ has a unique nonforking extension over Ab.

The next definition formalizes our idea of "sorts of types", mentioned above.

Recall that r' is a copy of r iff $r' = f(r)$ for some $f \in \mathrm{Aut}(\mathfrak{C})$. In Definition 2.5 we define an equivalence relation $\simeq$ on P^*.

2.5. **Definition.** (1) Let $r, r' \in P^*$. Then $r \simeq r'$ iff r is ACL$'$-interdependent with a copy of r'.

(2) For $n \leq n_a$ let $S_n = \{r/\simeq : r \in P^* \text{ and } n(r) = n\}$ and let $S = \bigcup_{0<n\leq n_a} S_n$. We call the $\simeq$-class of r the sort of r.

As we mentioned above, we want to find a nice essentially ACL$'$-closed $X \subseteq P^*$ with $\mathrm{ACL}'(X) = P^*$. In Definition 2.5 we define $\simeq$ so that if $r \simeq r'$ then in order to have an $r'' \in X$ with $r \in \mathrm{ACL}'(r'')$ we may include into X a copy of r'. In fact in the proof of Theorem 2.2 we could do well without sorts from S_n, $n < n_a$. But I think they are interesting in themselves, and have properties parallel to the sorts from S_{n_a}. In the next lemma we collect basic properties of $\simeq$.

2.6. **Lemma.** (1) *Suppose* $\underline{a}, \underline{b}, E \subseteq Q$, $\underline{a} \perp E$, $\underline{a} \perp E$, $\underline{a}, \underline{b}$ *are interalgebraic over* E *and* $r \in P^*_{\underline{b}}$. *Then for some copy* $r' \in P^*_{\underline{a}}$ *of* r, r *and* r' *are* ACL-*interdependent over* P^*_E.

(2) (*an alternative definition of* $\simeq$). *For* $r, r' \in P^*$, $r \simeq r'$ *iff* $n(r) = n(r')$ *and for some* $E \subseteq Q$ *independent from* $\pi(r)$, *for some copy* r'' *of* r', r *and* r'' *are* ACL-*interdependent over* P^*_E.

(3) $S_0 = \mathrm{ACL}(P^*_\varnothing)/\simeq$ *and all types in* $\mathrm{ACL}(P^*_\varnothing)$ *are* $\simeq$-*equivalent.*

(4) $S_1, \ldots, S_{n_a}$ *are nonempty and disjoint.*

(5) *Assume* $A \subseteq Q$ *is independent,* $|A| \geq n$. *Then* $S_n = \{r/\simeq : r \in P^*_A \text{ and } n(r) = n\}$.

Proof. (1) W.l.o.g. both $\underline{a}$ and $\underline{b}$ are independent, $n = n(r) = |\underline{b}| > 0$. Choose an independent n-tuple $\underline{d} \subseteq Q$ with $\underline{d} \perp \underline{ab}E$, and let $\underline{c} = \underline{d} + \underline{b}$. By Theorem 1.16, $P^*_{\underline{b}} \subseteq \mathrm{ACL}(P^*_{\underline{c}} P^*_{\underline{d}})$, hence by modularity of ACL, there are $r_{\underline{c}} \in P^*_{\underline{c}}$, $r_{\underline{d}} \in P^*_{\underline{d}}$ with $r \in \mathrm{ACL}(r_{\underline{c}}, r_{\underline{d}})$, and $r, r_{\underline{c}}, r_{\underline{d}}$ are pairwise ACL-independent. Since $\underline{b} \subseteq \mathrm{acl}(\underline{a}E)$, by [H] there are $\underline{a}' \subseteq \mathrm{acl}(\underline{a}) \cap G$, $\underline{e} \subseteq \mathrm{acl}(E) \cap G$ with $\underline{b} = \underline{a}' + \underline{e}$. Since $\underline{a}, \underline{b}$ are interalgebraic over E, $\underline{a}' \subseteq Q$ and $\underline{a}, \underline{a}'$ are interalgebraic over E, hence $\underline{a}'$ is independent. Choose $f \in \mathrm{Aut}(\mathfrak{C})$ fixing $\underline{d}$ and $r_{\underline{d}}$ and sending $\underline{b}$ to $\underline{a}'$. Let $r' = f(r)$, $\underline{c}' = f(\underline{c})$ and $r_{\underline{c}'} = f(r_{\underline{c}})$. Hence $\underline{c}' = \underline{d} + \underline{a}'$, $r_{\underline{d}} \in \mathrm{ACL}(r', r_{\underline{c}'})$. We have $\underline{c} \perp \underline{ab}E$ and $\underline{c} - \underline{c}' = \underline{b} - \underline{a}' = \underline{e} \subseteq \mathrm{acl}(E)$, hence $\underline{cc}' \perp \underline{ab}(E)$. We have $r \in \mathrm{ACL}(r_{\underline{c}} r_{\underline{d}}) \subseteq \mathrm{ACL}(r_{\underline{c}} r' r_{\underline{c}'})$. $\underline{cc}' \perp \underline{ab}(E)$ implies $P^*_{\underline{ab}} \perp P^*_{\underline{cc}'}(P^*_E)$, hence since $r' \in P^*_{\underline{a}}$, we get r and r' are ACL-interdependent over P^*_E.

(2) $\rightarrow$ is clear. $\leftarrow$. Let $A = \pi(r)$, $A'' = \pi(r'')$. Since r, r'' are ACL-interdependent over P^*_E, A and A'' are interalgebraic over E, hence $\dim(A/E) = \dim(A''/E)$. Also, $\dim(A) = \dim(A'') = n(r) = n(r'')$, hence $A \perp E$ gives $A'' \perp E$. By (1) there is a copy $r^* \in P^*_A$ of r'' such that r^*, r'' are ACL-interdependent over P^*_E, hence also r and r^* are ACL-interdependent over P^*_E. Now $E \perp A$ implies r, r^* are ACL$'$-interdependent.

(3), (4), (5) are trivial.

We shall use the next lemma in a many-model argument to show that S is finite.

2.7. **Lemma.** *Assume* $0 < n \leq n_a$, $R = \{r_i, i \in I\} \subseteq P^*$, $n(r_i) = n$, $\{\pi(r_i), i \in I\}$ *is independent,* $r \in P^*$ *is* ACL$'$-*interdependent with* r_t *over* $R' = R \backslash \{r_t\}$ *for some* $t \in I$, *and* $n(r) \leq n$. *Then* $n(r) = n$ *and* $r \simeq r_t$.

Proof. Let $A_i = \pi(r_i)$, $B = \pi(r)$, $A' = \bigcup\{A_i: i \in I\backslash\{t\}\}$. We have B and A_t are interalgebraic over A' and $A_t \perp A'$, hence also $B \perp A'$ and $n(r) = n(r_t) = n$. By Lemma 2.6(2) with $E := A'$, $r \simeq r_t$, so we are done.

2.8. **Theorem.** *S is finite. Moreover, $|S_{n_a}| = 1$.*

Proof. Let $n > 0$. First we prove that S_n is finite. Suppose not. Choose $R = \{r_i, i < \omega\} \subseteq P^*$ with $n(r_i) = n$ and $\{r_i, i < \omega\}$ pairwise $\simeq$-nonequivalent. We can assume $\{\pi(r_i), i < \omega\}$ is independent. For $X \subseteq \omega$ we define R_X as $\{r_i, i \in X\}$. By Lemma 2.7, if $r \in \mathrm{ACL}'(R_X)\backslash \mathrm{ACL}'(\varnothing)$ and $n(r) \leq n$, then for some $i \in X$, $r \simeq r_i$. It follows that for $X \neq Y \subseteq \omega$, $\mathrm{ACL}'(R_x)$ and $\mathrm{ACL}'(R_Y)$ are nonisomorphic. This shows $I(T, \aleph_0) = 2^{\aleph_0}$, a contradiction.

Now suppose $r, r' \in P^*$, $n(r) = n(r') = n_a$. W.l.o.g. $\pi(r), \pi(r')$ are independent. Choose $r'' \in \mathrm{ACL}'(rr')$ such that r, r', r'' are pairwise ACL'-independent. By Lemma 2.7 we see that $r'' \simeq r$ and $r'' \simeq r'$. This shows $r \simeq r'$, hence $|S_{n_a}| = 1$.

Let $\mathscr{s}$ be the only sort in S_{n_a}. Now we are going to choose for any independent $B \subseteq Q$ with size $n \leq n_w$ and for $s \in S_n$, a representative of s in P_B^*, uniformly in B. That is, for $B' \subseteq Q$ with $B'' \equiv B$, the representatives of s in P_B^* and $P_{B'}^*$ should be conjugate. But the notion of conjugate is somewhat ambiguous. If $r \in S(B)$ and $f: B \to B'$ is any bijection, then f is elementary, and we might consider $f(r)$ a conjugate of r over B'. The picture is clarified if we fix enumerations of B and B' and insist that f preserves these enumerations. Then f is unique, hence also $f(r)$ is unique. This is the reason for the notation we introduce now.

For $n \leq n_a$ let $Q^{(n)}$ be the set of independent n-tuples of elements of Q, and we stipulate $Q = Q^{(1)}$. If $\underline{a}, \underline{a}' \in Q^{(n)}$, r is a strong type over $\underline{a}$, r' is a strong type over $\underline{a}$, r' is a strong type over $\underline{a}'$, then we say that r and r' are conjugate if for b, b' realizing r, r' respectively, we have $\underline{a}b \equiv \underline{a}'b'$. Let $0 < n \leq n_a$ and $s \in S_n$. For every $\underline{a} \in Q^{(n)}$ we choose $s_{\underline{a}} \in P_{\underline{a}}^*$ so that $s_{\underline{a}}/\simeq = s$ and for $\underline{a}, \underline{a}' \in Q^{(n)}$, $s_{\underline{a}}$ and $s_{\underline{a}'}$ are conjugate. Moreover, let $s^* \in S_1$ be the $\simeq$-class of some (any) $r \in P_a^0$, $a \in Q$. We choose s_a^* as a stationarization of p_a. Let b realize $s_{\underline{a}}$. Hence $\mathrm{Mlt}(b/\underline{a})$ is finite and $\mathrm{Mlt}(\underline{a}) = 1$, and so $\mathrm{Mlt}(\underline{a}b)$ is finite. It follows that for some $E_s \in FE(\varnothing)$, if $c = \underline{a}b/E$ then $c \in \mathrm{acl}(\varnothing)$ and $\mathrm{tp}(\underline{a}b/c)$ is stationary. We add names for the elements of the finite set C of all E_s-classes, $s \in S$, to the signature. The next lemma shows that this does not really affect $\simeq$.

2.9. **Lemma.** *If we define, according to Definition 2.5, $\simeq_C$ and S^C in the new signature, then for $0 < n \leq n_a$ and $\underline{a} \in Q^{(n)}$, for every $s_C \in S_n^C$ there is an $s \in S_n$ and a conjugate r of $s_{\underline{a}}$ over $\underline{a}$ such that $r/\simeq_C = s_C$.*

Proof. By Lemma 2.6(5), in $P_{\underline{a}}^*$ there are representatives of all sorts in S_n^C. Suppose $r \in P_{\underline{a}}^*$, $r/\simeq_C = s_C$. For some $s \in S_n$, $r \simeq s_{\underline{a}}$, hence for some copy r' of $s_{\underline{a}}$, r and r' are ACL'-interdependent. It follows that $r \simeq_C r'$. Obviously, for some $\underline{a}' \in Q^{(n)}$ interalgebraic with $\underline{a}$, r' is a conjugate over $\underline{a}'$ of $s_{\underline{a}'}$. Choose $f \in \mathrm{Aut}_C(\mathfrak{C})$ sending $\underline{a}'$ to $\underline{a}$. Let $r'' = f(r')$. Clearly, $r \simeq_C r''$, and r'' is a conjugate over $\underline{a}$ of $s_{\underline{a}}$, as required.

We can assume now that the original $\simeq$ and S have been defined after adding

C to the signature, and by Lemma 2.10 we stipulate the following assumption.

2.10. **Assumption.** If $0 < n \leq n_a$, $s \in S_n$, $\underline{a} \in Q^{(n)}$ and b realizes $s_{\underline{a}}$ then $\mathrm{tp}(\underline{a}b)$ is stationary.

The next lemma shows that all conjugates over $\underline{a}$ of $s_{\underline{a}}$ are ACL$'$-interdependent. The original author's proof of this lemma consisted in constructing arbitrarily large almost full sets. Here we give a simplified version of this proof, suggested by S. Buechler. The proof is similar to that of Lemma 2.6(1).

2.11. **Lemma.** *Suppose* $0 < n \leq n_a$, $\underline{a}, \underline{a}' \in Q^{(n)}$, $s \in S_n$, r *is a conjugate over* $\underline{a}$ *of* $s_{\underline{a}}$ *and* r' *is a conjugate over* $\underline{a}'$ *of* $s_{\underline{a}'}$. *Assume* $E \subseteq Q$, $\underline{a} \perp E$, $\underline{a}' \perp E$ *and* $\underline{a} - \underline{a}' \subseteq \mathrm{acl}(E)$ (*the subtraction is pointwise, in* G). *Then* r *and* r' *are* ACL*-interdependent over* P^*_E.

Proof. Choose $\underline{b} \in Q^{(n)}$ independent from $\underline{a}E$, and let $\underline{c} = \underline{a} + \underline{b}$ (the addition is pointwise). Hence $\underline{c} \in Q^{(n)}$ and $\underline{a} \perp \underline{c}(E)$. By Theorem 1.16 and modularity of ACL, there are $r_{\underline{b}} \in P^*_{\underline{b}}$ and $r_{\underline{c}} \in P^*_{\underline{c}}$ such that $r \in \mathrm{ACL}(r_{\underline{b}} r_{\underline{c}})$ and $r, r_{\underline{b}}, r_{\underline{c}}$ are pairwise ACL-independent. By Fact 2.4 and Assumption 2.10 there is an $f \in \mathrm{Aut}(\mathfrak{C})$ fixing E pointwise, with $f(r_{\underline{b}}) = r_{\underline{b}}$, $f(\underline{a}) = \underline{a}'$ and $f(r) = r'$. Let $\underline{c}' = f(\underline{c})$ and $r_{\underline{c}'} = f(r_{\underline{c}})$. So we have $\underline{c}' = \underline{a} + \underline{b}$. $\underline{a} - \underline{a}' \subseteq \mathrm{acl}(E)$ implies $\underline{c} - \underline{c}' \subseteq \mathrm{acl}(E)$, hence $\underline{a} \perp \underline{c}(E)$ gives $\underline{aa}' \perp \underline{cc}'(E)$. We have also $r' \in \mathrm{ACL}(r_{\underline{c}'} r_{\underline{b}})$. Since $r_{\underline{b}} \in \mathrm{ACL}(r r_{\underline{c}})$, $r' \in \mathrm{ACL}(r r_{\underline{c}} r_{\underline{c}'})$. Thus $\underline{aa}' \perp \underline{cc}'(E)$ gives $rr' \perp r_{\underline{c}} r_{\underline{c}'}(P^*_E)$. It follows that r and r' are ACL-interdependent over P^*_E.

In particular, applying Lemma 2.11 to the case $\underline{a} = \underline{a}'$, $E = \varnothing$, we see that all conjugates of $s_{\underline{a}}$ over $\underline{a}$ are ACL$'$-interdependent. The next remark shows that we have achieved our goal of choosing a nice essentially ACL$'$-closed subset X of P^* with $\mathrm{ACL}'(X) = P^*$.

2.12. *Remark.* Let $X = \{s_{\underline{a}} : s \in S_n, n > 0, \underline{a} \in Q^{(n)}\}$. Then X is essentially ACL$'$-closed and $\mathrm{ACL}'(X) = P^*$.

Proof. Let $r \in P^*$. It suffices to prove that for some $r' \in X$, $r \in \mathrm{ACL}'(r')$. So we may assume $n(r) > 0$. By the definition of $s_{\underline{a}}$, for some $s \in S_n$ and $\underline{a} \in Q^{(n)}$, $r \simeq s_{\underline{a}}$. That is r is ACL$'$-interdependent with a copy of $s_{\underline{a}}$ over some $\underline{a}' \in Q^{(n)}$. By Lemma 2.11, any copy of $s_{\underline{a}}$ over $\underline{a}'$ is ACL$'$-interdependent with $s_{\underline{a}'}$. Hence $r \in \mathrm{ACL}'(s'_{\underline{a}})$.

Now we shall investigate ACL$'$-dependence on X from Remark 2.12. The next theorem is a generalization of [Bu4, 3.14]. Theorem 3.14 in [Bu4] deals with a single $s \in S$ (namely with s^*, corresponding to p_a), while we deal here with all $s \in S$ at once. Also we give a different proof.

2.13. **Theorem.** *Let* $s \in S_n$, $n > 0$. *Then there is a division subring* F_s *of the ring of matrices* $M_{n\times n}(F_q)$ *such that if* $A \cup \{\underline{a}\} \subseteq Q^{(n)}$ *and* $\underline{a}_{\mathscr{A}} \in F_s$-span$(A_{\mathscr{A}})$ *then* $s_{\underline{a}} \in \mathrm{ACL}'(\{s_{\underline{b}} : \underline{b} \in A\})$. *Moreover, if* A *is independent, then the converse is true, that is for* $\underline{a} \in Q^{(n)}$, $s_{\underline{a}} \in \mathrm{ACL}'(\{s_{\underline{b}} : \underline{b} \in A\})$ *iff* $\underline{a}_{\mathscr{A}} \in F_s$-span$(A_{\mathscr{A}})$.

Proof. It is easy to define F_s: let $F^*_s = \{\alpha \in M^*_{n\times n}(F_q) :$ for $\underline{a}, \underline{b} \in Q^{(n)}$ with $\underline{b}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha$, $s_{\underline{b}} \in \mathrm{ACL}'(s_{\underline{a}})\}$, and let $F_s = F^*_s \cup \{0\}$. By this definition F^*_s is a multiplicative subgroup of $M^*_{n\times n}(F_q)$. So if we prove that F_s is also an additive subgroup of $M_{n\times n}(F_q)$, then the theorem will be proved for $|A| = 1$.

The key to the proof will be however the case $|A| = 2$. Choose $\underline{a}, \underline{b} \in Q^{(n)}$ with $\underline{a} \perp \underline{b}$. Let $H = \{(\alpha, \beta): \alpha, \beta \in M^*_{n\times n}(F_q) \cup \{\mathbf{0}\}, (\alpha, \beta) \neq (\mathbf{0}, \mathbf{0})$ and for $\underline{c} \in Q^{(n)}$ with $\underline{c}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha + \underline{b}_{\mathscr{A}}\beta$, $s_{\underline{c}} \in \mathrm{ACL}'(s_{\underline{a}}, s_{\underline{b}})\} \cup \{(\mathbf{0}, \mathbf{0})\}$.

2.14. **Claim.** (1) $(\alpha, \beta) \in H \to (\beta, \alpha) \in H$.
(2) $(\alpha, \beta) \in H$, $\gamma \in F_s \to (\gamma\alpha, \beta) \in H$.
(3) $\gamma \in F_s \to (\mathbf{0}, \gamma) \in H$.
(4) $(\alpha, \beta) \in H \to \beta \in F_s$.
(5) $H = F_s \times F_s$.

Proof. (1)–(3) are obvious. (4). W.l.o.g. $\beta \neq \mathbf{0}$. We have $s_{\underline{c}} \in \mathrm{ACL}'(s_{\underline{a}}, s_{\underline{b}})$, where $\underline{c}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha + \underline{b}_{\mathscr{A}}\beta$. Since β is invertible, $\underline{c} \perp \underline{a}$, hence by Lemma 2.11 (with $E = \underline{a}$), if $\underline{b}' \in Q^{(n)}$ and $\underline{b}'_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\beta$ then $s_{\underline{c}}$ and $s_{\underline{b}'}$ are ACL-interdependent over $P^*_{\underline{a}}$. Hence $s_{\underline{b}'} \in \mathrm{ACL}(s_{\underline{b}}P^*_{\underline{a}})$. But $\underline{a} \perp \underline{b}\underline{b}'$, hence $s_{\underline{b}'} \in \mathrm{ACL}'(s_{\underline{b}})$, giving $\beta \in F_s$.

(5) By (1)–(4) it suffices to prove that for some $\alpha, \beta \in M^*_{n\times n}(F_q)$, $(\alpha, \beta) \in H$. Let $\underline{c} = \underline{a} + \underline{b}$. By Theorem 1.16 there are $r \in P^*_{\underline{a}}$ and $r' \in P^*_{\underline{b}}$ such that $s_{\underline{c}} \in \mathrm{ACL}(r, r')$. By Lemma 2.6(2), $r \simeq r' \simeq s_{\underline{c}}$, hence there are $\alpha, \beta \in M^*_{n\times n}(F_q)$ such that if $\underline{a}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha$, $\underline{b}'_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\beta$ then $r \in \mathrm{ACL}'(s_{\underline{a}'})$, $r' \in \mathrm{ACL}'(s_{\underline{b}'})$. Thus $\underline{c}_{\mathscr{A}} = \underline{a}'_{\mathscr{A}}\alpha^{-1} + \underline{b}'_{\mathscr{A}}\beta^{-1}$ and $s_{\underline{c}} \in \mathrm{ACL}'(s_{\underline{a}'}s_{\underline{b}'})$. This shows $(\alpha^{-1}, \beta^{-1}) \in H$.

In particular, if $\underline{c}_{\mathscr{A}} = \underline{a}_{\mathscr{A}} + \underline{b}_{\mathscr{A}}$ then $s_{\underline{c}} \in \mathrm{ACL}'(s_{\underline{a}}s_{\underline{b}})$. By the exchange principle, $s_{\underline{b}} \in \mathrm{ACL}'(s_{\underline{a}}s_{\underline{c}})$ and $\underline{b}_{\mathscr{A}} = \underline{c}_{\mathscr{A}} - \underline{a}_{\mathscr{A}}$, hence $(1, -1) \in H$, and $-1 \in F_s$. So to prove that F_s is a division subring of $M_{n\times n}(F_q)$ it suffices to prove that F_s is closed under $+$. Let $\alpha, \beta \in F_s$, $\underline{c}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha + \underline{b}_{\mathscr{A}}$ and $\underline{c}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\beta + \underline{c}_{\mathscr{A}}$. $(\alpha, 1), (\beta, 1) \in H$, hence $s_{\underline{c}'} \in \mathrm{ACL}'(s_{\underline{a}}s_{\underline{c}}) \subseteq \mathrm{ACL}'(s_{\underline{a}}s_{\underline{b}})$. But $\underline{c}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}(\alpha + \beta) + \underline{b}_{\mathscr{A}}$, i.e. $(\alpha + \beta, 1) \in H$. Hence $\alpha + \beta \in F_s$. Now we prove that for $A \subseteq Q^{(n)}$ and $\underline{a} \in Q^{(n)}$,

$$\text{(6)} \qquad \text{if } \underline{a}_{\mathscr{A}} \in F_s\text{-span}(A_{\mathscr{A}}) \text{ then } s_{\underline{a}} \in \mathrm{ACL}'(\{s_{\underline{b}}: \underline{b} \in A\}).$$

W.l.o.g. A is finite. First suppose A is independent. Then (6) follows by an easy induction, by Claim 2.14(5). Now we approach the general case. Let $A = \{\underline{a}^1, \ldots, \underline{a}^k\}$ for some k, and choose $\underline{c}^1, \ldots, \underline{c}^k \in Q^{(n)}$ such that $\{A, \underline{c}^1, \ldots, \underline{c}^k\}$ is independent. Let $C = \{\underline{c}^1, \cdots, \underline{c}^k\}$. Suppose $\underline{a} \in Q^{(n)}$ and $\underline{a}_{\mathscr{A}} = \sum \underline{a}^i_{\mathscr{A}}\alpha^i$, $\alpha^i \in F^*_s$. Let $\underline{b}^i = \underline{a}^i + \underline{c}^i$, and choose $\underline{b} \in Q^{(n)}$ with $\underline{b}_{\mathscr{A}} = \sum \underline{b}^i_{\mathscr{A}}\alpha^i$. Notice that $B = \{\underline{b}^1, \ldots, \underline{b}^k\}$ is independent, hence we get $s_{\underline{b}} \in \mathrm{ACL}'(s_{\underline{b}^1} \cdots s_{\underline{b}^k})$. Also, $\underline{a}^i, \underline{b}^i, \underline{c}^i$ are pairwise independent for each i, hence $s_{\underline{b}^i} \in \mathrm{ACL}'(s_{\underline{a}^i}, s_{\underline{c}^i})$, and $s_{\underline{a}^i}$. $s_{\underline{b}^i}$ are ACL-interdependent over P^*_C. Similarly, $s_{\underline{a}}$ and $s_{\underline{b}}$ are ACL-interdependent over P^*_C. Hence we get $s_{\underline{a}} \in \mathrm{ACL}(s_{\underline{a}^1}, \ldots, s_{\underline{a}^k}, P^*_C)$. Since $A \perp C$, we get $s_{\underline{a}} \in \mathrm{ACL}'(s_{\underline{a}^1}, \ldots, s_{\underline{a}^k})$.

To finish the proof, suppose $A \subseteq Q^{(n)}$ is independent, $\underline{a} \in Q^{(n)}$ and $s_{\underline{a}} \in \mathrm{ACL}'(\{s_{\underline{b}}: \underline{b} \in A\})$. We want to prove $\underline{a}_{\mathscr{A}} \in F_s$-span$(A_{\mathscr{A}})$. We may assume A is finite, and proceed by induction on $|A|$. If $|A| = 1$ we are done by the definition of F_s. Suppose $\underline{b} \in A$, $A' = A \setminus \{\underline{b}\}$, and $s_{\underline{a}}, s_{\underline{b}}$ are ACL$'$-interdependent over $\{s_{\underline{c}}: \underline{c} \in A'\}$. It follows that $\underline{a} \perp A'$ and $\underline{a}, \underline{b}$ are interalgebraic over A'. Hence there are $\underline{a}' \subseteq \mathrm{acl}(\underline{b})$, $\underline{a}'' \subseteq \mathrm{acl}(A')$ with $\underline{a} = \underline{a}' + \underline{a}''$. $\underline{a} \perp A'$ implies $\underline{a}' \in Q^{(n)}$, hence for some $\gamma \in M^*_{n\times n}(F_q)$, $\underline{a}'_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\gamma$. By Lemma 2.11, $s_{\underline{a}}$ and $s_{\underline{a}'}$ are ACL-interdependent over P^*_A, hence $s_{\underline{a}'} \in \mathrm{ACL}(s_{\underline{b}}P^*_{A'})$, and since

$\underline{b} \perp A'$, $s_{\underline{a}'} \in \mathrm{ACL}'(s_{\underline{b}})$ and $\gamma \in F_s$. We may suppose that for every $\underline{b}' \in A$, $s_{\underline{a}}$ and $s_{\underline{b}'}$ are ACL$'$-interdependent over $\{s_{\underline{c}} : \underline{c} \in A \backslash \{\underline{b}'\}\}$. Let $\underline{b}' \in A'$. By the same argument as above, $\underline{a} \perp A\backslash\{\underline{b}'\}$, hence $\underline{a} \perp \underline{b}$. It follows that $\underline{a}'' \in Q^{(n)}$. We have $\underline{a}'' = \underline{a} - \underline{a}'$, hence $s_{\underline{a}''} \in \mathrm{ACL}'(s_{\underline{a}} s_{\underline{a}'}) \subseteq \mathrm{ACL}'(\{s_{\underline{c}} : \underline{c} \in A\})$, and since $\underline{a}'' \subseteq \mathrm{acl}(A')$, $s_{\underline{a}''} \in \mathrm{ACL}'(\{s_{\underline{c}} : \underline{c} \in A'\})$. By the inductive hypothesis, $\underline{a}''_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A'_{\mathscr{A}})$. Together we get $\underline{a}_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\gamma + \underline{a}''_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$. Theorem 2.13 describes to some extent ACL$'$-dependence on the set $X_s = \{s_{\underline{a}} : \underline{a} \in Q^{(n)}\}$ (for $s \in S_n$). If we could waive in the "moreover" part the assumption that A is independent, then the description would be full, ACL$'$ would be modular on X_s, and X_s with ACL$'$-dependence would be projective over F_s. Notice however that even this would not imply $F_s \cong F_p$, because X_s is not necessarily essentially ACL$'$-closed. When $s \in S_1$ then Theorem 2.13 gives F_s, a division subring of F_q, and if $A \subseteq Q$, $a_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$ and $a \in Q$ then $s_a \in \mathrm{ACL}'(\{s_b : b \in A\})$. But if A is dependent over $\varnothing$, there may be other $a \in \mathrm{acl}(A) \cap Q$ with $a_{\mathscr{A}} \notin F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$ and yet $s_a \in \mathrm{ACL}'(\{s_b : b \in A\})$. When $s \in S_n$, $n > 1$, we encounter an additional difficulty. Namely, for $A \subseteq Q^{(n)}$ there may be an n-tuple $\underline{a} \in G^n \backslash Q^{(n)}$ with $\underline{a}_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$. Then we cannot conclude that $s_{\underline{a}} \in \mathrm{ACL}'(\{s_{\underline{b}}, \underline{b} \in A\})$ just because $s_{\underline{a}}$ is not defined! Defining suitably $s_{\underline{a}}$ by "projecting types" will be the main remaining trick in the proof of Theorem 2.2. This will be done in the next section. Notice yet that F_s is preserved by $\mathrm{Aut}(\mathfrak{C})$.

3. Projecting types

In this section we conclude the proof of Theorem 2.2. First we show that for $s \in S$, F_s may be regarded as a division subring of F_p (hence F_s is a field), and in fact F_s is isomorphic to F_p, which gives an embedding $i: F_p \to M_{n_a \times n_a}(F_q)$ such that $i(F_p)$ is a division subring of $M_{n_a \times n_a}(F_q)$. Then, projecting types, we extend the definition of $s_{\underline{a}}$ to all $\underline{a} \in G^{n_a} \backslash \mathrm{acl}(\varnothing)$ so that a counterpart of Theorem 2.13 holds. This gives a correspondence between P^* and G^{n_a} satisfying all the conditions of Theorem 2.2, except possibly for (5). We fulfill this last condition by a suitable change of coordinates. At the end of this section we give a hint on how to prove Theorem 2.2$'$. We begin with a closer look at some special $R \subseteq P^*$.

3.1. **Lemma.** *Assume $s \in S_n$, $n > 0$, $A \subseteq Q^{(n)}$ is independent. Let $R = \mathrm{ACL}'(\{s_{\underline{a}}, \underline{a} \in A\})$.*

(1) *Assume $r \in R$ and $0 < n(r) \leq n$. Then $n(r) = n$ and for some $\underline{b} \in Q^{(n)}$, $\underline{b}_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$ and r, $s_{\underline{b}}$ are ACL$'$-interdependent.*

(2) *Suppose $\underline{a}^0, \dots, \underline{a}^k \in Q^{(n)}$, $\underline{a}^i_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$ for $i \leq k$. Then the following conditions are equivalent.*

(a) *$\underline{a}^0, \dots, \underline{a}^k$ are independent.*

(b) *$\underline{a}^0_{\mathscr{A}}, \dots, \underline{a}^k_{\mathscr{A}}$ are linearly independent over F_s.*

(c) *$s_{\underline{a}^0}, \dots, s_{\underline{a}^k}$ are ACL$'$-independent*

Proof. (1) By Lemmas 2.7 and 2.11, r and $s_{\underline{b}}$ are ACL$'$-interdependent for some $\underline{b} \in Q^{(n)}$. By Theorem 2.13, $\underline{b}_{\mathscr{A}} \in F_s\text{-}\mathrm{span}(A_{\mathscr{A}})$.

(2) (b) $\to$ (a) is an easy exercise from linear algebra. W.l.o.g. A is finite. Suppose $\underline{a}^0_{\mathscr{A}}, \dots, \underline{a}^k_{\mathscr{A}}$ are F_s-linearly independent. By induction on $i \leq k+1$

we find sets A_i, $i \leq k+1$, with $A = A_0 \supsetneq A_1 \supsetneq \cdots \supsetneq A_{k+1}$ such that for every $i \leq k$, $A'_{i+1} \cup \{\underline{a}^0, \ldots, \underline{a}^i\}$ is independent and F_s-span$(A_{\mathscr{A}}) = F_s$-span(A'_{i+1}). Suppose we have A_i and want to find A_{i+1}.

By inductive hypothesis, $\underline{a}^i_{\mathscr{A}} \in F_s$-span$(A'_{i\mathscr{A}})$. Since A'_i is independent, $\underline{a}^i_{\mathscr{A}}$ can be presented in a unique way as an F_s-linear combination of elements of $A'_{i\mathscr{A}}$. As $\underline{a}^0_{\mathscr{A}}, \ldots, \underline{a}^k_{\mathscr{A}}$ are F_s-linearly independent, there must be an $\underline{a} \in A_i$ such that $\underline{a}_{\mathscr{A}}$ and $\underline{a}^i_{\mathscr{A}}$ are F_s-dependent over $A'_{i\mathscr{A}} \setminus \{\underline{a}_{\mathscr{A}}\}$. Let $A_{i+1} = A_i \setminus \{\underline{a}\}$. We see that A_{i+1} satisfies our requirements. For $i = k+1$ we get that $\{\underline{a}^0, \ldots, \underline{a}^k\} \subseteq A'_{k+1}$ is independent.

(a) $\to$ (c) follows by Corollary 1.4, (c) $\to$ (b) follows from Theorem 2.13. Using lemma 3.1 we get an embedding of F_s into F_p.

3.2. **Lemma.** *For every $s \in S$ there is a ring monomorphism $i_s: F_s \to F_p$. $i_{\mathfrak{s}}$ is an isomorphism.*

Proof. Let $s \in S_n$, $n > 0$. Choose independent $\underline{a}, \underline{b}, \underline{c} \in Q^{(n)}$, let $R = \mathrm{ACL}'(s_{\underline{a}}, s_{\underline{b}}, s_{\underline{c}})$ and $X_s = \{r \in R: r = s_{\underline{d}}$ for some $\underline{d} \in Q^{(n)}\}$. Hence after identifying ACL$'$-interdependent types, R becomes a projective plane over F_p, and by Lemma 3.1, X_s with ACL$'$-dependence becomes a projective plane over F_s. Consider the line ℓ in R generated by $s_{\underline{a}}, s_{\underline{b}}$. If we call $s_{\underline{a}}$ ∞, $s_{\underline{b}}$ 0 and $s_{\underline{a}+\underline{b}}$ 1, then the well-known algorithm endows $\ell \setminus \{\infty\}$ with a structure of a division ring isomorphic to F_p. The same algorithm, restricted to $\ell_{X_s} = \ell \cap X_s$ endows $\ell_{X_s} \setminus \{\infty\}$ with a structure of a division ring isomorphic to F_s. So $\ell_{X_s} \setminus \{\infty\}$ becomes a division subring of $\ell \setminus \{\infty\}$, which gives the required $i_s: F_s \to F_p$. In case when $s = \mathfrak{s}$, by Lemma 3.1(1), $X_{\mathfrak{s}}$ is essentially ACL$'$-closed and $\mathrm{ACL}'(X_{\mathfrak{s}}) = R$, hence $\ell = \ell \cap X_{\mathfrak{s}}$. This means that $i_{\mathfrak{s}}$ is an isomorphism.

Let $i = i_{\mathfrak{s}}^{-1}$. Hence $i: F_p \to M_{n_a \times n_a}(F_q)$, as required in Theorem 2.2. We shall see that we can regard P^* as a vector space over $F_{\mathfrak{s}}$ so that ACL$'$-dependence becomes just linear dependence over $F_{\mathfrak{s}}$. We shall define $\mathfrak{s}_{\underline{a}} \in P^*$ for any $\underline{a} \in G^{n_a} \setminus \mathrm{acl}(\varnothing)$. We would like to have always $\mathfrak{s}_{\underline{a}} \in P^*_{\underline{a}}$. For that reason, for $\underline{a} \in G^{n_a}$ define $P^*_{\underline{a}}$ as P^*_A, where $A = \mathrm{acl}(\underline{a}) \cap Q$. This generalizes naturally the convention introduced in §1.

3.3. **Definition.** Let $\underline{a} \in G^{n_a} \setminus \mathrm{acl}(\varnothing)$. Choose $\underline{d} \in Q^{(n_a)}$ with $\underline{d} \perp \underline{a}$. Let $\underline{c} = \underline{a} + \underline{d}$. So $\underline{c} \in Q^{(n_a)}$. We define $\mathfrak{s}_{\underline{a}}$ as any $r \in P^*_{\underline{a}}$ which is ACL-interdependent with $\mathfrak{s}_{\underline{c}}$ over $P^*_{\underline{d}}$. If $\underline{a} \in G^{(n_a)}$ and $\underline{a} \subseteq \mathscr{A}$, we can define $\mathfrak{s}_{\underline{a}}$ as any type in $P^*_{\varnothing}$, provided $P^*_{\varnothing}$ is nonempty (this case is really not important).

Let me explain the idea underlying Definition 3.3. If $\underline{a} \in Q^{(n_a)}$, we will see below that the new and the old $\mathfrak{s}_{\underline{a}}$ are ACL$'$-interdependent, so that Definition 3.3 extends the previous notation. For such an $\underline{a}$ we may assume the new $\mathfrak{s}_{\underline{a}}$ equals the old one. If $\underline{a}$ is not independent, we may think of $\mathfrak{s}_{\underline{a}}$ as a degenerated version of a "real" $\mathfrak{s}_{\underline{c}}$, a projection of $\mathfrak{s}_{\underline{c}}$. Suppose $\underline{c} \in Q^{(n_a)}$, $\underline{c} = (c_1, \ldots, c_{n_a})$, and we add c_1 to the signature. Then $\mathfrak{s}_{\underline{c}}$, in the new signature, is a degenerated version of $\mathfrak{s}_{\underline{c}}$ in the old signature, because $\underline{c}$ is no longer in $Q^{(n_a)}$. Still it bears some resemblance to the old $\mathfrak{s}_{\underline{c}}$. This is expressed in Definition 3.3. In the next lemma we prove some basic properties of $\mathfrak{s}_{\underline{a}}$, $\underline{a} \in G^{n_a}$.

3.4. **Lemma.** *Let* $\underline{a} \in G^{n_a}$, $\underline{a}_{\mathscr{A}} \neq \mathbf{0}$.

(1) $\mathscr{s}_{\underline{a}}$ *exists, moreover, up to* ACL$'$*-interdependence, the choice of* $\mathscr{s}_{\underline{a}}$ *is unique, hence Definition* 3.3 *is correct.*

(2) *If* $\underline{a} \in Q^{(n_a)}$ *then the new* $\mathscr{s}_{\underline{a}}$ *as defined in Definition* 3.3 *is* ACL$'$*-interdependent with the old* $\mathscr{s}_{\underline{a}}$, *defined after Theorem* 2.8, *hence we may assume that the new* $\mathscr{s}_{\underline{a}}$ *equals the old one in this case.*

(3) $\pi(\mathscr{s}_{\underline{a}}) = \operatorname{acl}(\underline{a}) \cap Q$, $n(\mathscr{s}_{\underline{a}}) = \dim(\underline{a})$.

(4) *For every* $r \in P^*$ *there is* $\underline{b} \in G^{n_a}$ *such that* r *and* $\mathscr{s}_{\underline{b}}$ *are* ACL$'$*-interdependent.*

(5) *Suppose* $\underline{d} \in Q^{(n_a)}$ *is independent from* $\underline{a}$, $\underline{c} = \underline{a} + \underline{d}$. *Then* $\mathscr{s}_{\underline{a}}, \mathscr{s}_{\underline{c}}, \mathscr{s}_{\underline{d}}$ *are pairwise* ACL$'$*-independent, and* $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\mathscr{s}_{\underline{c}}, \mathscr{s}_{\underline{d}})$.

Proof. (1) First we prove that $\mathscr{s}_{\underline{a}}$ exists. Choose $\underline{d} \in Q^{(n_a)}$ with $\underline{d} \perp \underline{a}$ and let $\underline{c} = \underline{a} + \underline{d}$. So $\underline{c} \in Q^{(n_a)}$ and $\underline{c} \perp \underline{a}$. By Theorem 1.16, $P^*_{\underline{c}} \subseteq \mathrm{ACL}(P^*_{\underline{a}} P^*_{\underline{d}})$, hence there is $r \in P^*_{\underline{a}}$ ACL-interdependent with $\mathscr{s}_{\underline{c}}$ over $P^*_{\underline{d}}$. Hence $\mathscr{s}_{\underline{a}}$ exists. If $r'' \in P^*_{\underline{a}}$ is another type ACL-interdependent with $\mathscr{s}_{\underline{c}}$ over $P^*_{\underline{d}}$, then $r \in \mathrm{ACL}(r'' P^*_{\underline{d}})$, hence $\underline{d} \perp \underline{a}$ implies $r \in \mathrm{ACL}'(r'')$. To prove that the choice of $\mathscr{s}_{\underline{a}}$ is unique up to ACL$'$-interdependence, choose $\underline{d}' \in Q^{(n_a)}$ independent from $\underline{ad}$, let $\underline{c}' = \underline{a} + \underline{d}'$, and suppose $r' \in P^*_{\underline{a}}$ is ACL-interdependent with $\mathscr{s}_{\underline{c}'}$ over $P^*_{\underline{d}'}$. By Assumption 2.10, there is an $f \in \mathrm{Aut}(\mathfrak{C})$ fixing $\underline{a}$ and r', with $f(c') = \underline{c}$ and $f(\mathscr{s}_{\underline{c}'}) = \mathscr{s}_{\underline{c}}$. Let $\underline{d}'' = f(\underline{d}')$. We see that $\underline{d}'' = f(\underline{c}') - \underline{a} = \underline{c} - \underline{a} = \underline{d}$, and we have that $f(r') = r'$ is ACL-interdependent with $\mathscr{s}_{\underline{c}}$ over $P^*_{\underline{d}}$. By the previous paragraph, r' is ACL$'$-interdependent with $\mathscr{s}_{\underline{a}}$.

(2) If $\underline{a} \in Q^{(n_a)}$ then by Theorem 2.13, the old $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\mathscr{s}_{\underline{c}} \mathscr{s}_{\underline{d}})$ because $\underline{a}_{\mathscr{A}} \in F_{\mathscr{s}}\text{-}\mathrm{span}(\underline{c}_{\mathscr{A}} \underline{d}_{\mathscr{A}})$, hence by (1) we are done.

(3) Let $\underline{c}, \underline{d}$ be as in Definition 3.3. Hence $\dim(\underline{c}/\underline{d}) = \dim(\underline{a})$. Since $\mathscr{s}_{\underline{c}}$ and $\mathscr{s}_{\underline{a}}$ are ACL-interdependent over $P^*_{\underline{d}}$,

$$\dim(\pi(\mathscr{s}_{\underline{c}})/\underline{d}) = \dim(\pi(\mathscr{s}_{\underline{a}})/\underline{d}) = \dim(\pi(\mathscr{s}_{\underline{a}})).$$

Since $\pi(\mathscr{s}_{\underline{c}}) = \operatorname{acl}(\underline{c}) \cap Q$, $\dim(\underline{c}/\underline{d}) = \dim(\pi(\mathscr{s}_{\underline{c}})/\underline{d})$. Hence $\dim(\pi(\mathscr{s}_{\underline{a}})) = \dim(\underline{a})$.

(4) W.l.o.g. $r \notin P^*_{\varnothing}$. Let $A = \pi(r)$. Choose $\underline{c} \in Q^{(n_a)}$ independent from A, let $r' = \mathscr{s}_{\underline{c}}$. Since P^* is projective over F_p, there is $r'' \in \mathrm{ACL}(rr')$ such that r, r', r'' are pairwise ACL-independent. Choose $D \subseteq \operatorname{acl}(A\underline{c})$ with $D = \pi(r'')$. r' and r'' are ACL-interdependent over $P^*_{\underline{a}}$, hence $\dim(c/A) = \dim(D/A)$. It follows that $\dim(D/A) = n_a$. Also, since $n(r'') \le n_a$, $\dim(D) \le n_a$. We conclude $D \perp A$ and $\dim(D) = n_a$. As $\underline{c} \subseteq \operatorname{acl}(AD)$, there are $\underline{b} \subseteq \operatorname{acl}(A) \cap G$, $\underline{d} \subseteq \operatorname{acl}(D) \cap G$ with $\underline{c} = \underline{b} + \underline{d}$. Since $\underline{c} \perp A$, $\underline{d} \in Q^{(n_a)}$ and $\underline{d} \perp \underline{b}$. Also, r is ACL-interdependent with $r' = \mathscr{s}_{\underline{c}}$ over $P^*_{\underline{d}} = P^*_D$. Hence r is ACL-interdependent with $\mathscr{s}_{\underline{b}}$ over $P^*_{\underline{d}}$. But $\underline{b} \subseteq \operatorname{acl}(A)$ and $A \perp \underline{d}$, hence r is ACL$'$-interdependent with $\mathscr{s}_{\underline{b}}$.

(5) By Lemma 2.11 (with $E := \underline{a}$), $\mathscr{s}_{\underline{c}}$ and $\mathscr{s}_{\underline{d}}$ are ACL-interdependent over $P^*_{\underline{a}}$, so there is an $r \in P^*_{\underline{a}}$ such that $r \in \mathrm{ACL}(\mathscr{s}_{\underline{c}} \mathscr{s}_{\underline{d}})$. But $\mathscr{s}_{\underline{a}}$ and $\mathscr{s}_{\underline{c}}$ are ACL-interdependent over $P^*_{\underline{d}}$, hence $r \in \mathrm{ACL}(\mathscr{s}_{\underline{a}} P^*_{\underline{d}})$. Now $P^*_{\underline{d}} \perp P^*_{\underline{a}}(P^*_{\varnothing})$ implies r and $\mathscr{s}_{\underline{a}}$ are ACL$'$-interdependent.

3.5. **Corollary.** *The set* $X = \{\mathscr{s}_{\underline{a}} : \underline{a} \in G^{n_a}\}$ *is essentially* ACL$'$*-closed and* $\mathrm{ACL}'(X) = P^*$.

The next theorem shows that ACL'-dependence on the set X from Corollary 3.5 is essentially a linear dependence over $F_{\mathscr{s}}$.

3.6. **Theorem.** *Assume* $A \subseteq G^{n_a}$ *and* $\underline{a} \in G^{n_a}$. *Then* $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\{\mathscr{s}_{\underline{b}}, \underline{b} \in A\})$ *iff* $\underline{a}_{\mathscr{A}} \in F_{\mathscr{s}}\text{-}\operatorname{span}(A_{\mathscr{A}})$.

Proof. Since $G^{n_a}_{\mathscr{A}}$ is a vector space over $F_{\mathscr{s}}$, it suffices to prove the theorem for $|A| \leq 2$.

$\leftarrow$. First we consider the case when $A = \{\underline{b}\}$. W.l.o.g. $\underline{a}_{\mathscr{A}} \neq \mathbf{0} \neq \underline{b}_{\mathscr{A}}$. Suppose $\alpha \in F_{\mathscr{s}}$ and $\underline{a}_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\alpha$. Thus $\alpha \neq \mathbf{0}$ and α is invertible. We want to prove tht $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\mathscr{s}_{\underline{b}})$. Choose $\underline{d}, \underline{d}' \in Q^{(n_a)}$ independent from $\underline{b}$ so that $\underline{d}'_{\mathscr{A}} = \underline{d}_{\mathscr{A}}\alpha$. Let $\underline{c} = \underline{b} + \underline{d}$, $\underline{c}' = \underline{a} + \underline{d}'$. By Corollary 1.4 it suffices to prove $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}(\mathscr{s}_{\underline{b}} P^*_{\underline{d}})$. By Definition 3.3, $\mathscr{s}_{\underline{c}} \in \mathrm{ACL}(\mathscr{s}_{\underline{b}} P^*_{\underline{d}})$ and $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}(\mathscr{s}_{\underline{c}'} P^*_{\underline{d}})$. Since $\underline{c}'_{\mathscr{A}} = \underline{c}_{\mathscr{A}}\alpha$ and $\underline{c} \in Q^{(n_a)}$, by Theorem 2.13, $\mathscr{s}_{\underline{c}'} \in \mathrm{ACL}'(\mathscr{s}_{\underline{c}})$. Hence we get $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}(\mathscr{s}_{\underline{b}} P^*_{\underline{d}})$, and $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\mathscr{s}_{\underline{b}})$. The case $|A| = 2$ is similar, we leave it to the reader.

$\rightarrow$. For a change, let us consider now the case $A = \{\underline{b}, \underline{c}\}$, with $\underline{b}, \underline{c} \in G^{n_a}$ and $\underline{b}_{\mathscr{A}}, \underline{c}_{\mathscr{A}}$ linearly independent over $F_{\mathscr{s}}$, the case $|A| = 1$ being similar. Suppose $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\mathscr{s}_{\underline{b}}\mathscr{s}_{\underline{c}})$ and $\mathscr{s}_{\underline{a}}, \mathscr{s}_{\underline{b}}, \mathscr{s}_{\underline{c}}$ are pairwise ACL'-independent. We want to find $\alpha, \beta \in F_{\mathscr{s}}$ such that $\underline{a}_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\alpha + \underline{c}_{\mathscr{A}}\beta$. Obviously, $\underline{a} \subseteq \operatorname{acl}(\underline{bc})$. Choose $\underline{d}, \underline{d}', \underline{d}'' \in Q^{(n_a)}$ with $\{\underline{d}, \underline{d}', \underline{d}'', \underline{bc}\}$ independent. Let $\underline{e} = \underline{c} + \underline{d}$, $\underline{e}' = \underline{a} + \underline{d}'$, $\underline{e}'' = \underline{b} + \underline{d}''$. Consider $X = \{\mathscr{s}_{\underline{b}}, \mathscr{s}_{\underline{c}}, \mathscr{s}_{\underline{d}}, \mathscr{s}_{\underline{d}'}, \mathscr{s}_{\underline{d}''}\}$. $\mathrm{DIM}(X/P^*_{\varnothing}) = 5$, and by Lemma 3.4(5), since $\mathscr{s}_{\underline{a}} \in \mathrm{ACL}'(\mathscr{s}_{\underline{b}}\mathscr{s}_{\underline{c}})$, we have $\mathscr{s}_{\underline{e}}, \mathscr{s}_{\underline{e}'}, \mathscr{s}_{\underline{e}''} \in \mathrm{ACL}'(X)$. Each of the sets $\{\underline{d}, \underline{d}', \underline{d}''\}$ and $\{\underline{e}, \underline{e}', \underline{e}''\}$ is independent. By modularity of ACL, there is $r \in \mathrm{ACL}'(\mathscr{s}_{\underline{d}}\mathscr{s}_{\underline{d}'}\mathscr{s}_{\underline{d}''}) \cap \mathrm{ACL}'(\mathscr{s}_{\underline{e}}\mathscr{s}_{\underline{e}'}\mathscr{s}_{\underline{e}''}) \setminus \mathrm{ACL}'(\varnothing)$. By Lemma 3.1, r is ACL'-interdependent with $\mathscr{s}_{\underline{g}}$ for some $\underline{g} \in Q^{(n_a)}$. By Theorem 2.13 we find $\alpha_i, \beta_i \in F_{\mathscr{s}}$ for $i < 3$ such that

$$\begin{aligned}\underline{g}_{\mathscr{A}} &= \underline{d}_{\mathscr{A}}\alpha_0 + \underline{d}'_{\mathscr{A}}\alpha_1 + \underline{d}''_{\mathscr{A}}\alpha_2 = \underline{e}_{\mathscr{A}}\beta_0 + \underline{e}'_{\mathscr{A}}\beta_1 + \underline{e}''_{\mathscr{A}}\beta_2 \\ &= \underline{d}_{\mathscr{A}}\beta_0 + \underline{d}'_{\mathscr{A}}\beta_1 + \underline{d}''_{\mathscr{A}}\beta_2 + \underline{c}_{\mathscr{A}}\beta_0 + \underline{a}_{\mathscr{A}}\beta_1 + \underline{b}_{\mathscr{A}}\beta_2.\end{aligned}$$

Comparing the second and the last step in this sequence of equalities, we see that $\alpha_i = \beta_i$, some α_i is $\neq \mathbf{0}$ and $\underline{c}_{\mathscr{A}}\beta_0 + \underline{a}_{\mathscr{A}}\beta_1 + \underline{b}_{\mathscr{A}}\beta_2 = \mathbf{0}$. But $\underline{b}_{\mathscr{A}}, \underline{c}_{\mathscr{A}}$ are linearly independent over $F_{\mathscr{s}}$, hence $\beta_1 = \mathbf{0}$ would imply $\beta_0 = \beta_2 = \mathbf{0}$, a contradiction. Thus $\beta_1 \neq \mathbf{0}$, i.e. $\underline{a}_{\mathscr{A}} = -\underline{c}_{\mathscr{A}}\beta_0\beta_1^{-1} - \underline{b}_{\mathscr{A}}\beta_2\beta_1^{-1}$. Hence $\alpha = -\beta_0\beta_1^{-1}$, $\beta = -\beta_2\beta_1^{-1}$ satisfy our demands. Now we shall relate types $s_{\underline{a}}$, $s \in S_n$, $\underline{a} \in Q^{(n)}$, $n > 0$, and division rings F_s with types $\mathscr{s}_{\underline{b}}$, $\underline{b} \in G^{n_a}$ and field $F_{\mathscr{s}}$. For $s \in S_n$, $n > 0$, and for an $\underline{a}^* \in Q^{(n)}$ we choose $\alpha_s \in M_{n \times n_a}(F_q)$ such that if $\underline{b}^* \in G^{n_a}$ and $\underline{b}^*_{\mathscr{A}} = \underline{a}^*_{\mathscr{A}}\alpha_s$, then $\mathscr{s}_{\underline{b}^*}$ and $s_{\underline{a}^*}$ are ACL'-interdependent. Clearly, for $s = \mathscr{s}$ we can take $\alpha_{\mathscr{s}} = I$.

3.7. **Proposition.** *Assume* $s \in S_n$, $n > 0$.

(1) α_s *has rank* n, *hence for some* $\beta_s \in M_{n_a \times n}(F_q)$, $\alpha_s\beta_s = I$.

(2) *For every* $\underline{a} \in Q^{(n)}$, *if* $\underline{b} \in G^{(n_a)}$ *and* $\underline{b}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha_s$ *then* $\mathscr{s}_{\underline{b}}$ *and* $s_{\underline{a}}$ *are* ACL'*-interdependent.*

(3) $\beta \in F_s$ *iff for some* $\gamma \in F_{\mathscr{s}}$, $\alpha_s\gamma = \beta\alpha_s$.

(4) *For every* $\gamma \in F_{\mathscr{s}}$ *there is at most one* $\beta \in M_{n \times n}(F_q)$ *with* $\alpha_s\gamma = \beta\alpha_s$.

(5) *By* (4) *we define a partial function* f_s *with* $\mathrm{Dom}(f_s) \subseteq F_{\mathscr{s}}$, $\mathrm{Rng}(f_s) \subseteq M_{n \times n}(F_q)$ *so that* $f_s(\gamma) = \beta$ *iff* $\alpha_s\gamma = \beta\alpha_s$. *Then* $F'_s = \mathrm{Dom}(f_s)$ *is a division subring of* $F_{\mathscr{s}}$ *and* $f_s\colon F'_s \to F_s$ *is an isomorphism.*

Proof. (1) We know that $\mathscr{s}_{\underline{b}^*}$ and $s_{\underline{a}^*}$ are ACL$'$-interdependent. By Lemma 3.5(3) and the definition of $s_{\underline{a}^*}$, $\dim(\underline{b}^*) = n(\mathscr{s}_{\underline{b}^*}) = n(s_{\underline{a}^*}) = \dim(\underline{a}^*) = n$. Since $\underline{b}^*_{\mathscr{A}} = \underline{a}^*_{\mathscr{A}}\alpha_s$, rank of α_s equals n.

(2) Let f be an automorphism of $\mathfrak{C}$ fixing α_s, sending $\underline{a}$ to $\underline{a}^*$ and $\underline{b}$ to $\underline{b}'$. Let $r = f(s_{\underline{a}})$, $r' = f(\mathscr{s}_{\underline{b}})$. We have $\underline{b}' - \underline{b}^* \subseteq \mathscr{A}$, hence by Theorem 3.6 and Lemma 2.11, $r \in \mathrm{ACL}'(s_{\underline{a}^*})$ and $r' \in \mathrm{ACL}'(\mathscr{s}_{\underline{b}^*})$. It follows that r and r' are ACL$'$-interdependent, hence also $s_{\underline{a}}$ and $\mathscr{s}_{\underline{b}}$ are ACL$'$-interdependent.

(3) We neglect the case $\beta = \mathbf{0} = \gamma$. By Theorems 2.13 and 3.6, we have $\beta \in F_s^*$ iff for $\underline{a}, \underline{b} \in Q^{(n)}$ with $\underline{b}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\beta$, $s_{\underline{b}} \in \mathrm{ACL}'(s_{\underline{a}})$ iff for $\underline{a}', \underline{b}' \in G^{n_a}$ with $\underline{a}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha_s$, $\underline{b}'_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\alpha_s$, $\mathscr{s}_{\underline{b}'} \in \mathrm{ACL}'(\mathscr{s}_{\underline{a}'})$ iff for some $\gamma \in F_{\mathscr{s}}^*$, $\underline{b}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha_s\gamma$ iff for some $\gamma \in F_{\mathscr{s}}^*$, $\underline{a}_{\mathscr{A}}\beta\alpha_s = \underline{a}_{\mathscr{A}}\alpha_s\gamma$ iff for some $\gamma \in F_{\mathscr{s}}^*$, $\beta\alpha_s = \alpha_s\gamma$.

(4) If $\beta\alpha_s = \beta'\alpha_s$ for some $\beta, \beta' \in M_{n\times n}(F_q)$, then multiplying this equality on the right by β_s we get $\beta = \beta'$.

(5) It is easy to check that f_s is a homomorphism, and by (3), $\mathrm{Rng}(f_s) = F_s$. Multiplying the equation $\beta\alpha_s = \alpha_s\gamma$ on both sides by β^{-1} on the left and γ^{-1} on the right, we get $\beta^{-1}\alpha_s = \alpha_s\gamma^{-1}$. This shows that $f_s(\gamma) = \beta$ implies $f_s(\gamma^{-1}) = \beta^{-1}$. Hence $f_s\colon F_s' \to F_s$ is an epimorphism. Since F_s', F_s are division rings, f_s is an isomorphism of F_s' and F_s.

One obstacle prevents us from concluding the proof of Theorem 2.2. Namely it may happen that for $s = s^*$ (= the sort of p_a), $\alpha_s \neq (1, \mathbf{0})$. To deal with this difficulty we shall uniformly change coordinates. Recall that we have chosen $\mathscr{s}_{\underline{a}}$ for $\underline{a} \in Q^{(n_a)}$ in a rather arbitrary way, as a uniform representative in $P^*_{\underline{a}}$ of the sort $\mathscr{s}$. We could well choose at the beginning another $\mathscr{s}_{\underline{a}}$ to serve the same purpose. So now let $\alpha \in M_{n_a\times n_a}(F_q)$ be invertible. For $\underline{a} \in Q^{(n_a)}$ define $\mathscr{s}'_{\underline{a}}$ as $\mathscr{s}_{\underline{a}'}$, where $\underline{a}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha$ (by Lemma 2.11 this definition is correct, up to ACL$'$-interdependence). We shall see below how the change of coordinates affects $F_{\mathscr{s}}$ and α_s, $s \in S$. Let $F_{\mathscr{s}'} = \{\beta \in M^*_{n_a\times n_a}(F_q) \colon \text{for } \underline{a}, \underline{c} \in Q^{(n_a)} \text{ with } \underline{c}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\beta,\ \mathscr{s}'_{\underline{c}} \in \mathrm{ACL}'(\mathscr{s}'_{\underline{a}})\} \cup \{\mathbf{0}\}$.

3.8. **Lemma.** (1) *$\beta \in F_{\mathscr{s}'}$ iff $\alpha^{-1}\beta\alpha \in F_{\mathscr{s}}$, hence $F_{\mathscr{s}'}$ is the α^{-1}-conjugate of $F_{\mathscr{s}}$.*

(2) *For $s \in S_n$, $n > 0$, if $\alpha_s' = \alpha_s\alpha^{-1}$, $\underline{a} \in Q^{(n)}$, $\underline{b} \in G^{n_a}$ and $\underline{b}_A = \underline{a}_{\mathscr{A}}\alpha_s'$, then $\mathscr{s}'_{\underline{b}}$ and $s_{\underline{a}}$ are* ACL$'$-*interdependent.*

Proof. (1) We have $\beta \in F^*_{\mathscr{s}'}$ iff for $\underline{a}, \underline{b} \in Q^{(n_a)}$ with $\underline{b}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\beta$, $\mathscr{s}'_{\underline{b}} \in \mathrm{ACL}'(\mathscr{s}'_{\underline{a}})$ iff for $\underline{a}', \underline{b}' \in Q^{(n_a)}$ with $\underline{a}'_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha$, $\underline{b}'_{\mathscr{A}}\alpha$, $\mathscr{s}_{\underline{b}'} \in \mathrm{ACL}'(\mathscr{s}_{\underline{a}'})$ iff for some $\gamma \in F^*_{\mathscr{s}}$, $\underline{b}'_{\mathscr{A}} = \underline{a}'_{\mathscr{A}}\gamma$ iff for some $\gamma \in F^*_{\mathscr{s}}$, $\beta\alpha = \alpha\gamma$ iff $\alpha^{-1}\beta\alpha \in F^*_{\mathscr{s}}$.

(2) Let $s \in S_n$, $\underline{a} \in Q^{(n)}$, $\underline{b} \in G^{n_a}$ and $\underline{b}_{\mathscr{A}} = \underline{a}_{\mathscr{A}}\alpha_s$. Then $\mathscr{s}_{\underline{b}}$ and $s_{\underline{a}}$ are ACL$'$-interdependent. Choose $\underline{d} \in Q$ independent from $\underline{a}, \underline{b}$, let $\underline{c} = \underline{b}+\underline{d}$. By Definition 3.3, $\mathscr{s}_{\underline{b}}$ and $\mathscr{s}_{\underline{c}}$ are ACL-interdependent over $P^*_{\underline{d}}$. Choose $\underline{c}', \underline{d}' \in Q^{(n_a)}$ with $\underline{c}'_{\mathscr{A}}\alpha = \underline{c}_{\mathscr{A}}$, $\underline{d}'_{\mathscr{A}}\alpha = \underline{d}_{\mathscr{A}}$. Let $\underline{b}' = \underline{c}' - \underline{d}'$. Then also $\underline{b}'_{\mathscr{A}}\alpha = \underline{b}_{\mathscr{A}}$. We see that $\mathscr{s}'_{\underline{c}'}$ and $\mathscr{s}_{\underline{c}}$ are ACL$'$-interdependent, hence $\mathscr{s}'_{\underline{c}'}$ and $\mathscr{s}_{\underline{b}}$ are ACL$'$-interdependent over $P^*_{\underline{d}}$, and so $\mathscr{s}_{\underline{b}}$ and $\mathscr{s}'_{\underline{b}'}$ are ACL$'$-interdependent. We have $\underline{b}'_{\mathscr{A}} = \underline{b}_{\mathscr{A}}\alpha^{-1} = \underline{a}_{\mathscr{A}}\alpha_s\alpha^{-1} = \underline{a}_{\mathscr{A}}\alpha_s'$, so we are done.

We can choose α so that $\alpha_{s^*}\alpha^{-1} = (1, \mathbf{0})$. Hence we can assume that the choice of the original $\mathscr{s}_{\underline{a}}$ is such that the following holds.

3.9. **Assumption.** For $a \in Q$, s_a^* is ACL$'$-interdependent with $\mathfrak{s}_{\underline{a}}$, where $\underline{a} = (a, 0)$.

Now we can conclude the proof of Theorem 2.2. We define Φ as follows. For $r \in P^*$ and $\underline{a} \in G^{n_a}$, $\Phi(r, \underline{a})$ holds iff

$$(*) \qquad r \text{ and } \mathfrak{s}_{\underline{a}} \text{ are ACL}'\text{-interdependent.}$$

Φ is invariant under automorphisms of $\mathfrak{C}$, because $(*)$ is. $\mathrm{Dom}(\Phi) = P^*$ and $G^{n_a}\setminus\{0\} \subseteq \mathrm{Rng}(\Phi)$ by Lemma 3.4 and Definition 3.3. Condition (3) in Theorem 2.2 follows from Theorem 3.6. Condition (4) is obvious, (5) follows from Assumption 3.9.

Theorem 2.2$'$ may be proved as follows. One way is to repeat all the proof of Theorem 2.2, but with P^* replaced by P. Another way consists in noticing that in S_{n_b} there is a unique sort of types from P. Call this sort $\mathfrak{s}^0$. Since P is essentially ACL$'$-closed in P^*, P with ACL$'$-dependence is projective over F_p as well, hence $i_{\mathfrak{s}^0}$ from Lemma 3.2 is an isomorphism of $F_{\mathfrak{s}^0}$ and F_p. This gives an embedding $i'\colon F_p \to M_{n_b \times n_b}(F_q)$, as required in Theorem 2.2$'$. Then define types $\mathfrak{s}_{\underline{a}}^0$ for $\underline{a} \in G^{n_b}$, as projections of types $\mathfrak{s}_{\underline{c}}^0$, $\underline{c} \in Q^{(n_b)}$, similarly as in Definition 3.3. The rest is much the same as the proof of Theorem 2.2.

Now we shall draw some more corollaries from Theorems 2.2, 2.2$'$ and their proofs. F and F' are as in Theorems 2.2 and 2.2$'$. So F and F' are isomorphic copies of F_p. First, using Lemma 2.11 we improve Proposition 1.15 a little.

3.10. **Proposition.** *For every* $A \subseteq Q$, $P_A \subseteq \mathrm{ACL}(P_A^0 P_\varnothing) \subseteq \mathrm{ACL}'(P_A^0)$.

Proof. Notice that $P \perp P_\varnothing^*(P_\varnothing)$. We will tacitly use this in the proof. By Theorem 1.16 we may assume $A = \{a\}$. Choose an infinite $E \subseteq Q$ with $E \perp a$. Clearly $P_E \subseteq \mathrm{ACL}(P_E^0)$ and $P_{Ea} \subseteq \mathrm{ACL}(P_{Ea}^0)$. Suppose $r \in P_a$. We can assume that we have added to the language a sufficiently large finite subset of $\mathrm{acl}(\varnothing)$. So there are $a_1, \dots, a_k \in \mathrm{acl}(Ea) \cap Q \setminus \mathrm{acl}(E)$ (for some k), such that $r \in \mathrm{ACL}(s_{a_1}^* \cdots s_{a_k}^* P_E)$. $a_i \in \mathrm{acl}(Ea)\setminus \mathrm{acl}(E)$ implies there are $b_i \in \mathrm{acl}(a) \cap Q$ and $e_i \in \mathrm{acl}(E) \cap G$ such that $a_i = b_i + e_i$. By Lemma 2.11, $s_{a_i}^*$ and $s_{b_i}^*$ are ACL-interdependent over P_E^*, hence over P_E. Thus also $r \in \mathrm{ACL}(s_{b_1}^* \cdots s_{b_k}^* P_E)$. Now $E \perp a$ implies $P_E \perp P_a(P_\varnothing)$, hence $r \in \mathrm{ACL}(s_{b_1}^* \cdots s_{b_k}^* P_\varnothing)$, which finishes the proof.

In the next corollary we show various connections between F_p and F_q, following from Theorems 2.2, 2.2$'$ (or their proofs). In Theorem 1.23, using rough geometrical means, we proved that if $n_d > 1$ and F_q is finite, then F_p is finite as well. Here we get more: F_p is finite iff F_q is finite. Corollary 3.11(4) below was proved also in [Bu4, §4].

3.11. **Corollary.** (1) *The dimension of* $(F_q)^{n_a}$ *as a right vector space over* F *equals* n_d.

(2) *The dimension of* $(F_q)^{n_b}$ *over* F' *equals* n_e.

(3) F_p *is finite iff* F_q *is finite. In this case* $|F_p|^{n_d} = |F_q|^{n_a}$, $|F_p|^{n_e} = |F_q|^{n_b}$.

(4) F_p, F_q *have the same characteristic.*

(5) F_q-span *of the set* $\{(1, 0)\beta\colon \beta \in F'\}$ *is the whole of* $(F_q)^{n_b}$, *regarded as the left vector space over* F_q.

(6) $[F_p^* : F_{s^*}^*] \geq n_b$.

(7) *If* F_p, F_q *are finite then*

$$\frac{|F_p|^{n_d} - 1}{|F_p| - 1} = \sum_{s \in S_1} \frac{|F_q| - 1}{|F_s| - 1}.$$

Proof. (1) Let $a \in Q$ and let $V_a = \{a_{\mathscr{A}}\beta : \beta \in (F_q)^{n_a}\}$. V_a is an F-subspace of $G_{\mathscr{A}}^{n_a}$. By Theorem 2.2, $\mathrm{DIM}(P_a^*/P_\varnothing^*)$ equals the F-dimension of V_a. The action of F on V_a is isomorphic to the action of F on $(F_q)^{n_a}$.

(2) The same proof.

(3) By Theorem 2.2, F_p is embeddable into $M_{n_a \times n_a}(F_q)$, hence if F_q is finite then F_p is finite, too. Conversely, suppose F_p is finite. By (1), (2), $|F_q|^{n_a} = |F_p|^{n_d}$ and $|F_q|^{n_b} = |F_p|^{n_e}$, hence F_q is finite.

(4) I is the identity of F.

(5) Let $a \in Q$. By Proposition 3.10, $\mathrm{ACL}'(\{s_b^* : b_{\mathscr{A}} = a_{\mathscr{A}}\alpha, \alpha \in F_q^*\}) = \mathrm{ACL}'(P_a)$. By Theorem 2.2′ this means that the set $\{\alpha(1, \mathbf{0})\beta : \alpha \in F_q, \beta \in F'\}$ is the whole of $(F_q)^{n_b}$.

(6) This is an application of the proof of Theorem 2.2′. As in Proposition 3.7 we find there $F'_{s^*} \subseteq F'$ and an isomorphism $f_{s^*} : F'_{s^*} \to F_{s^*}$, $\alpha_{s^*} = (1, 0)$ by Assumption 3.9. By (5), $\{\alpha_{s^*}\beta, \beta \in F'\}$ are F_q-linear generators of $(F_q)^{n_b}$. $\dim_{F_q}(F_q)^{n_b} = n_b$. Now if for $\beta, \beta' \in F'^*$, $\beta'\beta^{-1} \in F'_{s^*}$, then by Proposition 3.7(5), for some $\alpha \in F_{s^*}^* \subseteq F_q$, $\alpha\alpha_{s^*} = \alpha_{s^*}\beta'\beta^{-1}$, which means $\alpha\alpha_{s^*}\beta = \alpha_{s^*}\beta'$, hence $\alpha_{s^*}\beta$ and $\alpha_{s^*}\beta'$ are F_q-dependent. Thus $[F'^* : F'^*_{s^*}] \geq n_b$. But F_p, F' and F_{s^*}, F'_{s^*} respectively are isomorphic and locally finite, hence we are done.

(7) Let $a \in Q$. After identification of ACL′-interdependent types, P_a^* becomes a projective space over F_p of dimension n_d, hence has $(|F_p|^{n_d} - 1)/(|F_p| - 1)$ points. For each $r \in P_a^*$ there is a unique $s \in S_1$ such that for some $b \in \mathrm{acl}(a) \cap Q$, $r \in \mathrm{ACL}'(s_b)$. For each $s \in S_1$, the number of pairwise ACL′-independent types s_b, $b \in \mathrm{acl}(a)$, equals $(|F_q| - 1)/(|F_s| - 1)$, so we are done.

In the next section we shall discuss in greater detail the algebraic problem of counting nonisomorphic F-subspaces of $G_{\mathscr{A}}^{n_a}$, to which in Theorem 2.2 we reduced the problem of counting nonisomorphic ACL′-closed subsets of P^*. In fact, S. Buechler has proved in [Bu4] that both F_p and F_q are locally finite fields. Hence we shall consider the case when K and L are fields with $L \subseteq M_{n \times n}(K)$ for some n. We shall discuss also the relevance of this problem to the original problem (P).

4. A problem from linear algebra and a problem of parameters

Assume K is a countable field (or even a locally finite field), $n^* < \omega$, $L \subseteq M_{n^* \times n^*}(K)$ is a division subring of $M_{n^* \times n^*}(K)$, and is a field in its own right. Assume V is a very large right vector space over K (that is V is a "monster" K-space). Before Theorem 2.2 we defined the notion of isomorphism between L-subspaces of V^{n^*}. Let $I(K, L)$ be the number of isomorphism types of countable L-subspaces of V^{n^*} and $I'(K, L)$ be the number of isomorphism types of countable L-subspaces W of V^{n^*} such that $W = L\text{-}\mathrm{span}(\{(a, \mathbf{0}) \in W : a \in V\})$. Assume Γ is a group of automorphisms of K such that each $\gamma \in \Gamma$ preserves L and the Γ-orbit of every element of K is finite. Then $I(K, L, \Gamma)$ denotes the number of Γ-isomorphism types of countable L-subspaces of V^{n^*}. Similarly define $I'(K, L, \Gamma)$. Notice that $I(K, L, \Gamma) \leq I(K, L)$ and $I'(K, L, \Gamma) \leq I'(K, L)$.

In [BN] we show that $I(K, L) < 2^{\aleph_0}$ implies $I(K, L) = \aleph_0$. It is not clear in general whether $I(K, L, \Gamma) < 2^{\aleph_0}$ implies $I(K, L, \Gamma) = \aleph_0$. However, in many cases it turns out that $I(K, L) = I(K, L, \Gamma)$. This is due to the fact that $I(K, L) = 2^{\aleph_0}$ usually implies the existence of $2^{\aleph_0}$ nonisomorphic L-subspaces W_α, $\alpha < 2^{\aleph_0}$, of V^{n^*} constructed using finitely many distinguished elements D of K. If we name these elements, then W_α, $\alpha < 2^{\aleph_0}$, become non-Γ'-isomorphic, where $\Gamma' = \{\gamma \in \Gamma: \gamma(\alpha) = \alpha \text{ for every } \alpha \in D\}$. Since $[\Gamma : \Gamma']$ is finite, every Γ-isomorphism class splits into finitely many Γ'-isomorphism classes, which gives $I(K, L, \Gamma) = 2^{\aleph_0}$.

On the other hand $I(K, L) = \aleph_0$ implies of course $I(K, L, \Gamma) = \aleph_0$. Hence $I(K, L, \Gamma)$ and $I(K, L)$ are often equal. We shall discuss these problems in [BN] in greater detail, relying on [DR]. Now notice that certainly $I(K, L, \Gamma) = I(K, L)$ for finite K. This is because for finite K, also Γ is finite, hence every Γ-isomorphism class splits into finitely many isomorphism classes. Corollaries 2.3 and 2.3′ yield the following remark. F and F' are as in Theorems 2.2, 2.2′.

4.1. *Remark.* (1) $I(F_q, F, \Gamma_q)$ equals the number of isomorphism types of ACL'-closed countably dimensional subsets of P^*.

(2) $I'(F_q, F, \Gamma_q) = I'(F_q, F', \Gamma_q)$ equals the number of isomorphism types of sets $\mathrm{ACL}'(\{r \in P^0: r \text{ is realized in } M\})$, where M is a countable model of T.

This remark shows the relevance of $I(F_q, F, \Gamma_q)$ to the original problem (P). Also, by Remark 4.1, $I(T, \aleph_0) < 2^{\aleph_0}$ implies $I(F_q, F, \Gamma_q)$, $I'(F_q, F, \Gamma_q)$, $I(F_q, F', \Gamma_q)$, $I'(F_q, F', \Gamma_q)$ are all $< 2^{\aleph_0}$. Even if calculating $I(F_q, F, \Gamma_q)$ does not yet immediately solve (P), it determines $I(p|E, \aleph_0)$ for some finite E. We say that E is a basis of $P^*_\varnothing$ if E is a selector from $\{r(\mathfrak{C}): r \in R\}$, where R is an ACL-basis of $P^*_\varnothing$. Notice that by Lemma 1.7 R is necessarily finite, hence E is finite, too.

4.2. **Theorem.** *Assume* $I(F_q, F, \Gamma_q) = \aleph_0$. *There is a finite set* $E = E' \cup E''$, *where* $E' \subseteq \mathrm{acl}(\varnothing)$ *and* E'' *is a basis of* $P^*_\varnothing$, *such that* $I(p|E, \aleph_0)$ *is countable.*

Proof. Let E' be a finite subset of $\mathrm{acl}(\varnothing)$ required by Theorem 2.2, and $\underline{e} = E''$ be a basis of $P^*_\varnothing$. W.l.o.g. add E' to the signature. Suppose $I(p|E, \aleph_0)$ is uncountable. Then there are ACL'-closed countably dimensional sets $R_\alpha \subseteq P^*$, $\alpha < \omega_1$, which are pairwise nonisomorphic over E, that is for $\alpha \neq \beta$ there is no $f \in \mathrm{Aut}_E(\mathfrak{C})$ with $f[R_\alpha] = R_\beta$. For $f \in \mathrm{Aut}(\mathfrak{C})$, $f[R_\alpha]$ is determined by $f' = f|\mathrm{acl} > (Q)$. By Remark 4.1 w.l.o.g. R_α, $\alpha < \omega_1$ are isomorphic. Choose for $\alpha > 0$ $f_\alpha \in \mathrm{Aut}(\mathfrak{C})$ such that $f_\alpha[R_\alpha] = R_0$. Let $\underline{e}_\alpha = f_\alpha(\underline{e})$. We get an $\alpha \neq \beta < \omega_1$ with $\underline{e}_\alpha \equiv \underline{e}_\beta(\mathrm{acl}(Q))$. Choose $f \in \mathrm{Aut}(\mathfrak{C})$ with $f|\mathrm{acl}(Q) = \mathrm{id}$, $f(\underline{e}_\alpha) = \underline{e}_\beta$. Then $f' = f_\beta^{-1} f f_\alpha \in \mathrm{Aut}_E(\mathfrak{C})$ and $f'[R_\alpha] = R_\beta$, a contradiction.

The problem of determining $I(K, L)$ is similar to the problem of counting ACL'-closed subsets of P^*. It may be instructive to note that many ideas from the proof of Theorem 2.2 have a counterpart here. It is so with sorts of types s and fields F_s. Let us review shortly how these notions appear in the purely algebraic setting. We assume now that $I(K, L) < 2^{\aleph_0}$. For a matrix α, $\mathrm{rk}(\alpha)$ denotes the rank of α.

Let $0 < n \leq n^*$. We denote $\simeq$ on $\{\alpha \in M_{n\times n^*}(K)\colon \mathrm{rk}(\alpha) = n\}$ by $\alpha \simeq \beta$ iff for some $\gamma \in L^*$ and $\delta \in M^*_{n\times n}(K)$, $\alpha\gamma = \delta\beta$. We define $S_n(K, L) = \{\alpha/\simeq\colon \mathrm{rk}(\alpha) = n\}$, $S(K, L) = \bigcup_{0<n\leq n^*} S_n(K, L)$. This corresponds to Definition 2.5. For $s \in S_n(K, L)$ choose $\beta_s \in M_{n\times n^*}(K)$ with $\beta_s/\simeq = s$ so that if $s^* = (1, \mathbf{0})/\simeq$ then $\beta_{s^*} = (1, \mathbf{0})$ and for $\mathscr{s} = I/\simeq$ (here $I \in M_{n^*\times n^*}(K)$), $\beta_{\mathscr{s}} = I$.

Now it is trivial to see that $S_{n^*}(K, L) = \{\mathscr{s}\}$. Let $V^{(n)}$ denote the set of K-independent n-tuples from V. For $s \in S_n(K, L)$ and $\underline{a} \in V^{(n)}$ define $s_{\underline{a}}$ as $\underline{a}\beta_s \in V^{n^*}$. Notice that for every $\underline{a} \in V^{n^*}$ with $\dim_K(\underline{a}) = n$ there is a unique $s \in S_n(K, L)$ such that for some $\underline{b} \in \pi'(\underline{a})^{(n)}$, $\underline{a}$ and $s_{\underline{b}}$ are L-interdependent. Recall that for $v = (v_1, \ldots, v_n) \in V^n$, $\pi'(v) = K\text{-}\mathrm{span}(v_1, \ldots, v_n)$. Now there is no more trouble with projecting types: for $\underline{a} \in V^{n^*}$ we define $\mathscr{s}_{\underline{a}}$ as $\underline{a}\beta_{\mathscr{s}} = \underline{a}$.

However the counterparts of Lemma 2.7, Theorems 2.8 and 2.13, Lemma 3.8 and Corollary 3.11 seem more interesting. The proofs are always parallel to the original ones, but easier, so we omit them.

4.3. **Proposition.** *Suppose* $s^i \in S_n(K, L)$, $i \in I$, $\{\underline{b}_i \in V^{(n)}, i \in I\}$ *is* K-*independent (that is* $\pi'(\underline{b}_i) \cap \pi'(\{\underline{b}_j\colon j \neq i\}) = \{\mathbf{0}\}$),

$$\underline{a} \in V^{n^*} \cap L\text{-}\mathrm{span}(\{s^i_{\underline{b}_i}, i \in I\})$$

and $\dim(\underline{a}) \leq n$. *Then* $\dim(\underline{a}) = n$ *and if* $\underline{a}$ *and* $s^i_{\underline{b}_i}$ *are L-interdependent over* $\{s^j_{\underline{b}_j}, j \neq i\}$, *then for some* $\underline{b} \in \pi'(\underline{a})^{(n)}$, $\underline{a}$ *and* $s^i_{\underline{b}_i}$ *are L-interdependent.*

4.4. **Corollary.** $S(K, L)$ *is finite. For* $s \in S_n(K, L)$, $F_s = \{\beta \in M^*_{n\times n}(K)\colon$ *for some* $\gamma \in L$, $\beta_s\gamma = \beta\beta_s\} \cup \{\mathbf{0}\}$, $F'_s = \{\gamma \in L;$ *for some* $\beta \in F_s$, $\beta_s\gamma = \beta\beta_s\}$.

4.5. **Theorem.** *Let* $s \in S_n(K, L)$. *For* $A \subseteq V^{(n)}$ *and* $\underline{a} \in V^{(n)}$, $\underline{a} \in F_s\text{-}\mathrm{span}(A)$ *implies* $s_{\underline{a}} \in L\text{-}\mathrm{span}(\{s_{\underline{b}}, \underline{b} \in A\})$. *Moreover, if A is K-independent, then the converse is true, that is for* $\underline{a} \in V^{(n)}$, $s_{\underline{a}} \in L\text{-}\mathrm{span}(\{s_{\underline{b}}\colon \underline{b} \in A\})$ *iff* $\underline{a} \in F_s\text{-}\mathrm{span}(A)$.

4.6. **Proposition.** F_s *and* F'_s *are isomorphic, the function* $\{\langle\beta, \gamma\rangle \in F_s \times F'_s\colon \beta_s\gamma = \beta\beta_s\}$ *is an isomorphism of* F_s *and* F'_s.

4.7. *Remark.* $n' = \dim_L(K^{n^*})$ is finite. If K is finite then

$$\frac{|L|^{n'} - 1}{|L| - 1} = \sum_{s\in S_1(K, L)} \frac{|K| - 1}{|F_s| - 1}.$$

In Definition 1.19 we defined the notion of free decomposition of an ACL′-closed subset of P^*. Correspondence Φ from Theorem 2.2 translates this notion into the notion of free decomposition of an L-subspace of V^{n^*}.

4.8. **Definition.** Assume W is an L-subspace of V^{n^k}. $\{W_i, i \in I\}$ is a free decomposition of W (in V) if

(1) W_i is an L-subspace of V^{n*}, $W_i \neq \{\mathbf{0}\}$,

(2) $\pi'(W) = \bigoplus\{\pi'(W_i), i \in I\}$, in particular $\{\pi'(W_i), i \in I\}$ is K-independent,

(3) $W_i = \pi'(W_i)^{n^*} \cap W$ and

(4) $W = L\text{-span}(\bigcup_i W_i)$.
We say that W is decomposable if there is a free decomposition $\{W_i, i \in I\}$ of W with $|I| \geq 2$. Otherwise we say that W is indecomposable.

4.9. *Remark.* Assume that there are finitely many isomorphism types of indecomposable L-subspaces of V^{n^*}, and every L-subspace W of V^{n^*} has a free decomposition into indecomposables. Then $I(K, L)$ is countable.

The following proposition corresponds to Remark 1.20 and Proposition 1.21.

4.10. **Proposition.** *Assume $n^* = 1$ and $n' = \dim_L(K^{n^*}) = 1$. Then every indecomposable L-subspace of $V^{n^*} = V$ is isomorphic to aK, $a \in V \backslash \{\mathbf{0}\}$, and every L-subspace of V has a free decomposition into indecomposables. Every L-subspace of V is a K-subspace of V.*

In [Bu4] S. Buechler proved that if $n^* = 1$, $n' = 2$ (that is $[K : L] = 2$), then $I(K, L)$ is countable. He proves there that in this case every L-subspace W of V has a free decomposition $\{W_i, i \in I\} \cup \{W_j, j \in J\}$ into indecomposables, such that $\dim_K(\pi'(W_i)) = \dim_K(\pi'(W_j)) = 1$ for every i, j, every W_i is of the form aK and W_j of the form aL for some $a \in V$.

Clearly the isomorphism type of W is determined by the powers of I and J, hence $I(K, L) = \aleph_0$ here. Notice also, that W uniquely determines $K\text{-span}(\bigcup_{i \in I} W_i)$. Indeed, $a \in K\text{-span}(\bigcup_i W_i)$ iff $aK \subseteq W$. Call $K\text{-span}(\bigcup_i W_i)$ the full part of W.

It is important to know that we can get the remaining part of the decomposition of W in a rather arbitrary way. Namely, for any choice of $\{W_j, j \in J\}$ such that $W_j = a_j L \subseteq W$, if $\{a_j, j \in J\}$ is K-independent over the full part of W and

$$\pi'(W) = K\text{-span}(\{\pi'(W_i), i \in I\} \cup \{a_j, j \in J\}),$$

then $\{W_i, i \in I\} \cup \{W_j, j \in J\}$ is a free decomposition of W.

In the case $n^* = 1$ and $[K : L] = n' \geq 4$ [Bu4], it is proved that $I(K, L) = 2^{\aleph_0}$ and $I(K, L, \Gamma) = 2^{\aleph_0}$. A proof is essentially contained also in [DR] (which uses different language). In [BN] we analyze more deeply the problem of determining $I(K, L)$ for various K, L. Now let us check the impact on (P) of the few cases we considered above. This is summarized in the following proposition. n_a and n_b play the role of n^* here, and n_d, n_e the role of n'.

4.11. **Proposition.** (1) *Assume $n_a = 1$. Then $n_d \leq 4$. Moreover, if $n_d = 1$ or 2, then $I(p, \aleph_0)$ is countable.*

(2) *Assume $n_b = 1$. Then $n_e \leq 4$. Moreover, if $n_e = 1$ or 2 then $I(p, \aleph_0)$ is countable.*

The rest of this paper is devoted to the proof of this proposition. Since parts (1) and (2) are similar, we shall concentrate on (1). We assume that we have added to the signature a sufficiently large finite subset of $\mathrm{acl}(\varnothing)$, as required in Theorem 2.2. Assume $n_a = 1$. If $n_d \geq 4$ then as mentioned above, $I(F_q, F, \Gamma_q) = 2^{\aleph_0}$. By Corollary 2.3, $I(T, \aleph_0) = 2^{\aleph_0}$, a contradiction. Hence $n_d < 4$. Now suppose $n_d = 1$ or 2. Theorem 4.2 gives a finite set E such that $I(p|E, \aleph_0)$ is countable. However, since E is not embeddable into every model of T, we cannot conclude immediately that $I(p, \aleph_0)$ is countable (this is the problem of parameters from the title of this section). To do this we have to prove in each case that there are countably many isomorphism types of

ACL-closed countably dimensional subsets of P^*. But we know by Corollary 2.3 that there are countably many isomorphism types of ACL$'$-closed countably dimensionals subsets of P^*.

We shall consider only the case $n_d = 2$, as the case $n_d = 1$ is similar and easier. Let us enumerate $\mathrm{acl}(\varnothing)$ as $\{b_n, n < \omega\}$, and let $\{p_i, i < \omega\}$ be an enumeration of all complete stationary types over finite subsets of $\mathrm{acl}(\varnothing)$, with p_i being over $\{b_n, n < i\}$. Suppose $\underline{r} = (r_1, \dots, r_n)$ is an ACL-independent tuple of types from P^*, $\underline{c} \subseteq Q$ is a basis of $\pi(\underline{r})$. We say that $(\underline{r}, \underline{c})$ is k-determined if whenever a_i realizes r_i and $\underline{a} = (a_1, \dots, a_n)$ then $\mathrm{tp}(\underline{ac}/\{b_i: i < k\})$ is stationary and parallel to p_i (for some $i < k$). Then we say that $(\underline{r}, \underline{a})$ corresponds to this p_i.

4.12. **Lemma.** *Assume $R \subseteq P^*$ is ACL-closed, $A = \pi(R)$ and either $P^*_A \subseteq \mathrm{ACL}'(R)$ or for no $a \in A$, $P^*_a \subseteq \mathrm{ACL}'(R)$. Then there is $k < \omega$ such that $R \subseteq \mathrm{ACL}'(R_k)$, where $R_k = \{\underline{r}$: for some $a \in A$, $\underline{r} \subseteq P^*_a \cap R$ is an ACL$'$-basis of $P^*_a \cap R$, $P^*_a \cap R \not\subseteq \mathrm{ACL}'(\varnothing)$ and $(\underline{r}, \underline{a})$ is k-determined$\}$.*

Proof. First let us consider the case when $P^*_A \subseteq \mathrm{ACL}'(R)$. Suppose the lemma is false. Then we can choose n_i, $i < \omega$, so that n_0 is the minimal k such that $R_k \not\subseteq \mathrm{ACL}'(\varnothing)$ and n_{i+1} is the minimal k such that $R_k \not\subseteq \mathrm{ACL}'(R_{n_i})$. Of course, $R \subseteq \bigcup_n R_n$.

Choose $\underline{r}_i \in R_{n_i} \setminus \mathrm{ACL}'(\bigcup_{k<n_i} R_k)$ and $a_i \in A$ such that $(\underline{r}_i, a_i)$ is n_i-determined. Let $R^0 = R \cap P^*_\varnothing$. Hence $R \perp P^*_\varnothing(R^0)$. For $X \subseteq \omega$ let $R(X) = \mathrm{ACL}(R^0 \cup \{\underline{r}_i: i \in X\})$. Notice that $R(X) \subseteq R$. We can recover the set $\{n_i: i \in X\}$ from $R(X)$ as follows.

Let $R(X)_k$ be defined just as R_k but with R replaced everywhere by $R(X)$. Define by induction a sequence m_i, $i < \omega$, so that m_0 is the minimal k such that $R(X)_k \not\subseteq \mathrm{ACL}'(\varnothing)$ and m_{i+1} is the minimal k such that $R(X)_k \not\subseteq \mathrm{ACL}'(R(X)_{m_i})$. We see that m_i, $i < \omega$, is an increasing enumeration of $\{n_i, i \in X\}$. This shows that if $X \neq Y \subseteq \omega$ then $R(X)$, $R(Y)$ are nonisomorphic. This gives $I(T, \aleph_0) = 2^{\aleph_0}$.

The case when for no $a \in A$, $P^*_a \subseteq \mathrm{ACL}'(R)$ is handled similarly, modulo the following claim, which follows from the discussion of the case $n^* = 1$, $n' = 2$ before Proposition 4.11.

4.13. **Claim.** Suppose $A' = \{a_i, i \in J\} \subseteq A$ is independent, $\underline{r}_i \subseteq P^*_{a_i}$ is an ACL$'$-basis of $P^*_{a_i} \cap R$ and $P^*_{a_i} \cap R \not\subseteq \mathrm{ACL}'(\varnothing)$. Then

$$P^*_{A'} \cap R \subseteq \mathrm{ACL}'(\{\underline{r}_i, i \in J\}).$$

Now we can finish the proof of Proposition 4.11(1) in case $n_a = 1$, $n_d = 2$. Let R be an ACL-closed subset of P^* with countable dimension, let $R^0 = R \cap P^*_\varnothing$, $A = \pi(R)$, $R' = \mathrm{ACL}'(R)$ and let $W = \Psi(R')$, Ψ being defined in Corollary 2.3. By the discussion before Proposition 4.11, there is a free decomposition $\{W_i, i \in I\} \cup \{W_j, j \in J\}$ of W, as described there. We can assume that for every $j \in J$, $\Psi^{-1}(W_j) \cap R \not\subseteq \mathrm{ACL}'(\varnothing)$. Let $W_I = F\text{-span}(\bigcup_{i\in I} W_i)$, $W_J = F\text{-span}(\bigcup_{j\in J} W_j)$, $R_I = \Psi^{-1}(W_I) \cap R$, $R_J = \Psi^{-1}(W_J) \cap R$. We have $R_I \perp R_J(R^0)$, $R = \mathrm{ACL}(R_I R_J R^0)$. Let $\underline{r}^0$ be an ACL-basis of R^0. By Lemma 4.12 choose $k < \omega$ such that R_I, R_J and $(\underline{r}^0, \varnothing)$ are k-determined. Then we can find k-determined $(\underline{r}_i, a_i)$, $i \in I$, $(\underline{r}_i, a_j)$, $j \in J$ such that

(1) for $i \in I$, $a_i \in \pi(R_I)$, $\underline{r}_i \subseteq P^*_{a_i} \cap R$ is an ACL$'$-basis of $P^*_{a_i} \cap R$,

(2) for $j \in J$, $a_j \in \pi(R_J)$, $P^*_{a_j} \cap R \not\subseteq \mathrm{ACL}'(\varnothing)$ and $\underline{r}_j \subseteq P^*_{a_j} \cap R$ is an ACL'-basis of $P^*_{a_j} \cap R$,

(3) $\{a_i, i \in I\} \cup \{a_j, j \in J\}$ is independent and $A \subseteq \mathrm{acl}(\{a_i, i \in I\} \cup \{a_j, j \in J\})$. For $t < k$ let $I_t = \{i \in I: (\underline{r}_i, a_i)$ corresponds to $p_t\}$, $J_t = \{j \in J: (\underline{r}_j, a_j)$ corresponds to $p_t\}$ and i_0 is the minimal i such that $(\underline{r}^0, \varnothing)$ corresponds to p_i. W.l.o.g. add $\{b_t, t < k\}$ to the signature. We see that the sequences $(|I_t|: t < k)$, $(|J_t|: t < k)$ and i_0 determine the isomorphism type of R.

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